Part I: Graphical Symplectic Algebra

Cole Comfort

Joint work with: Robert I. Booth, Titouan Carette

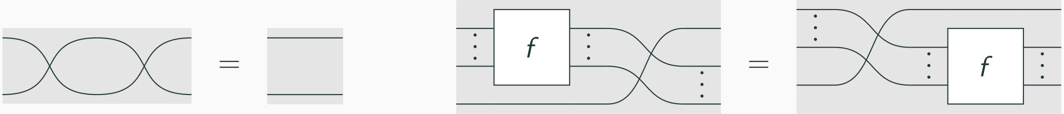
based on: arXiv:2401.07914

June 20, 2024

Motivation: I am bad at mathematics.

Props

A **prop** is a strict symmetric monoidal category generated by a single object...



A **compact prop** also allows for wires to be bent/unbent:

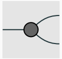
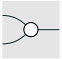
Graphical linear algebra

Affine matrices: generators

Given a field $\mathbb{K}$, finite dimensional affine transformations can be represented their **homogeneous coordinates matrices** (T, S are matrices, $\vec{a}, \vec{b}$ are vectors):

$$\left[\begin{array}{c|c} T & \vec{a} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} S & \vec{b} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} TS & T\vec{b}+\vec{a} \\ \hline 0 & 1 \end{array}\right]$$

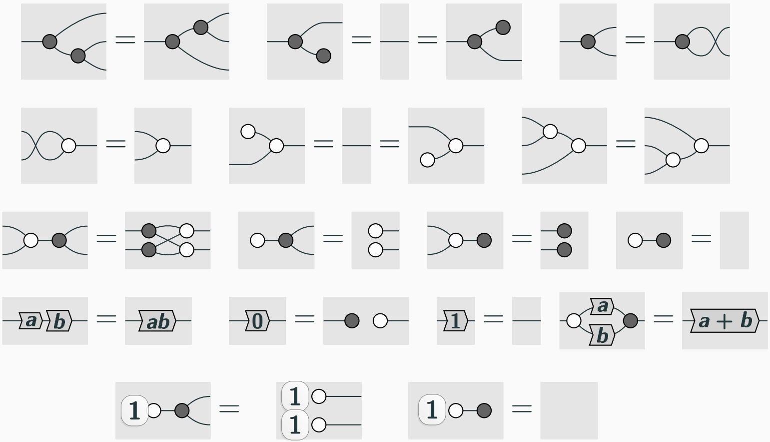
The prop of affine transformations between finite dimensional vector spaces is generated by the homogeneous coordinate matrices:

$\llbracket$  $\rrbracket = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ \hline & 1 \end{array}\right]$ $\llbracket$  $\rrbracket = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline & & 1 \end{array}\right]$ $\llbracket$  $\rrbracket = \left[\begin{array}{c|c} a & 0 \\ \hline & 1 \end{array}\right]$

$\llbracket$  $\rrbracket = \left[\begin{array}{c|c} * & * \\ \hline & 0 \end{array}\right]$ $\llbracket$  $\rrbracket = \left[\begin{array}{c|c} * & 0 \\ \hline & 0 \end{array}\right]$ $\llbracket$  $\rrbracket = \left[\begin{array}{c|c} * & 0 \\ \hline & 1 \end{array}\right]$

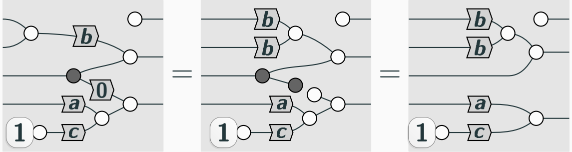
Affine matrices: axioms

Modulo the equations:

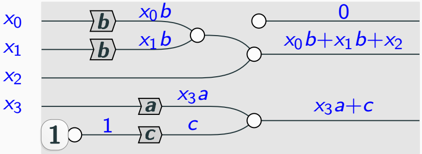


Example of matrix multiplication

The following diagram can be simplified to a normal form:



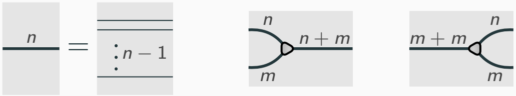
Following the paths from left to right gives us the homogeneous coordinate matrix:

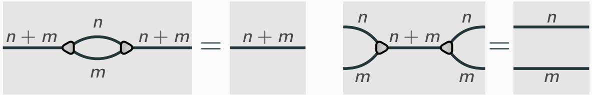


$$\rightsquigarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ b & b & 1 & 0 & 0 \\ 0 & 0 & 0 & a & c \\ \hline & & & & 1 \end{array}\right] \left[\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ x_0 b + x_1 b + x_2 \\ x_3 a + c \\ \hline 1 \end{array}\right]$$

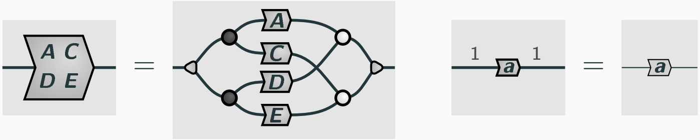
Strictification and block matrices

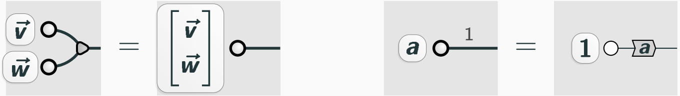
Every prop can be strictified to an $\mathbb{N}$-coloured prop:





This allows us to define block matrices/vectors diagrammatically:





Affine relations (Bonchi et al. [Bon+19],Bonchi et al. [BSZ17])

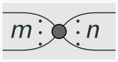
Given a field $\mathbb{K}$, the compact prop of $\mathbb{K}$-affine relations, $\mathsf{AffRel}_{\mathbb{K}}$, has:

- **Morphisms** $n \to m$ are affine subspaces $S \subseteq \mathbb{K}^n \oplus \mathbb{K}^m$.
- **Composition** relational, for $S : n \to m$, $T : m \to k$

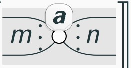
$$R \circ S := \{(\vec{x}, \vec{z}) \in \mathbb{K}^n \oplus \mathbb{K}^k \mid \exists \vec{y} \in \mathbb{K}^m : (\vec{x}, \vec{y}) \in S \quad \text{and} \quad (\vec{y}, \vec{z}) \in R\}$$

- **Symmetric monoidal structure** given by direct sum;
- **Compact structure** same as Rel.

$\mathsf{AffRel}_{\mathbb{K}}$ is generated by the following relations, for all $a \in \mathbb{K}$:

$\llbracket$  $\rrbracket$

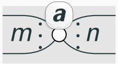
$$:= \left\{ \left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}, \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \right) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; a \in \mathbb{K} \right\}$$

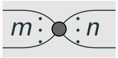
$\llbracket$  $\rrbracket$

$$:= \left\{ (\vec{b}, \vec{c}) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; \sum_{j=0}^{n-1} b_j + \sum_{k=0}^{m-1} c_k = a \right\}$$

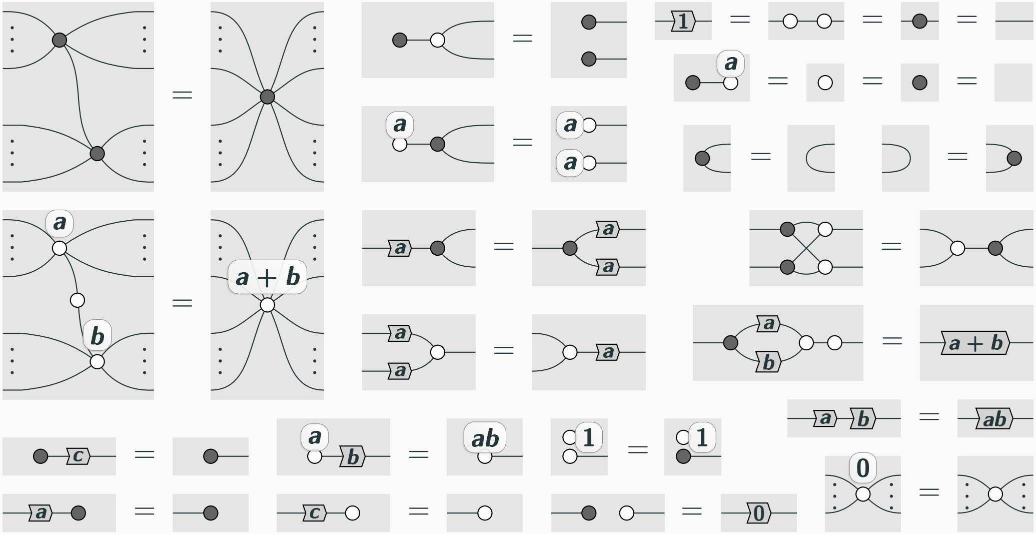
$\llbracket$  $\rrbracket$

$$:= \{(b, ab) \mid b \in \mathbb{K}\}$$

Modulo, the "spiders" 

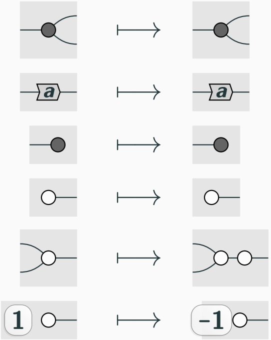
and 

being commutative, undirected and,



for all $a, b \in \mathbb{K}$, $c \in \mathbb{K}^{\times}$.

The embedding $\mathsf{AffMat}_{\mathbb{K}} \hookrightarrow \mathsf{AffRel}_{\mathbb{K}}$ taking an affine transformation $T : n \to m$ to it's graph $\{(\vec{x}, T\vec{x}) \mid \vec{x} \in \mathbb{K}^m\}$ sends:



Classical mechanics and symplectic geometry

Electrical circuits: current and voltage

The extensional behaviour of an electrical circuits is characterised by how it transforms current and voltage;

- **Ohm's law:** The voltage around the node in a circuit is equal to the current multiplied by the resistance.
- **Kirchhoff's current law:** The sum of currents flowing into a node is equal to the sum of currents flowing out of the node.

Example

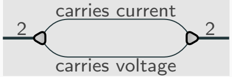
Given a linear resistor with resistance $r \in \mathbb{R}^{>0}$ on a wire with incoming current/votage (z_0, x_0) and outgoing current/voltage (z_1, x_1):

- by KCL, currents equalize: $z_0 = z_1$;
- by OL, the outgoing current becomes: $x_1 = x_0 + z_0 r$.

String diagrams for electrical circuits, take I

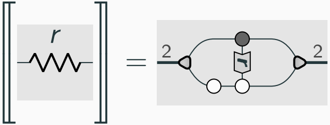
Following Baez et al. [BCR18] and Baez and Fong [BF18], we can represent electrical circuit components as real affine relations.

Using the string diagrams from Bonchi et al. [Bon+19], decompose a wire into a current and voltage



...the resistor is represented as follows:

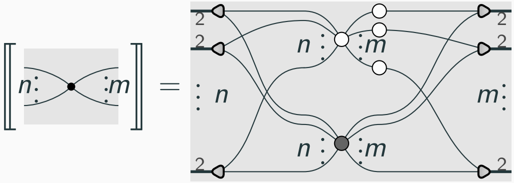
Example



More string diagrams for electrical circuits, take I

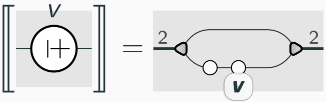
Example

Ideal wire junctions *sum currents*, and *equalize voltages*:



Example

Constant voltage source *does nothing to current* and *adds to the voltage*:



What is the more conceptual picture?

Classical mechanical systems can be represented by the configurations of abstract **positions** Z and **momenta** X:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure momentum	pressure
Thermal	entropy	entropy flow	temperature momentum	temperature

For n-particles in Euclidean space, *the space of possible configurations of positions/momenta* $\mathbb{R}^{2n} \cong \mathbb{R}^n_Z \oplus \mathbb{R}^n_X$ is the **phase space**.

Table adapted from Smith [Smi93, page 23, table 2.1] and Baez and Fong [BF18]

Affine Lagrangian subspaces

Definition
Two configurations $(\vec{z}, \vec{x}), (\vec{q}, \vec{p}) \in \mathbb{K}^{2n}$ of phase-space are **compatible** when:

$$\vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q} = 0$$

The bilinear map

$$\omega_n : \mathbb{K}^{2n} \oplus \mathbb{K}^{2n} \to \mathbb{K} \quad \left((\vec{z}, \vec{x})\ , (\vec{q}, \vec{p})\right) \mapsto \vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q}$$

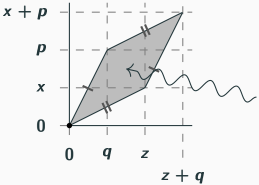
is a **symplectic form**, and the phase space $(\mathbb{K}^{2n}, \omega_n)$ is a **symplectic vector space**.

An **affine Lagrangian subspace** is a *maximally compatible* affine subspace of a symplectic vector space.

Remark (Baez and Fong [BF18], Baez et al. [BCR18])
Resistors, voltages sources and junctions of wires are affine Lagrangian subspaces.

Geometric interpretation of compatibility

Example
In the phase-space of a single particle, $(\mathbb{K}^2, \omega_1)$, the symplectic form measures area:



$$\omega_1\left(\begin{bmatrix} z \\ x \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}\right)$$

Compatible points are colinear, so affine Lagrangian subspaces are lines.

An affine Lagrangian subspaces don't represent single particle; but an ensemble of particles *flowing along a trajectory*.

Affine Lagrangian relations

Definition (Guillemin and Sternberg [GS79],Weinstein [Wei82])
The **compact prop of affine Lagrangian relations** $\mathsf{AffLagRel}_{\mathbb{K}}$ has:

- **Morphisms** $n \to m$, given by (possibly empty) affine Lagrangian subspaces of $(\mathbb{K}^{2n} \oplus \mathbb{K}^{2m}, \omega_n - \omega_m : \mathbb{K}^{2(n+m)} \oplus \mathbb{K}^{2(n+m)} \to \mathbb{K})$.
- **Composition** is given by relational composition.
- **Symmetric monoidal structure** is given by the direct sum.

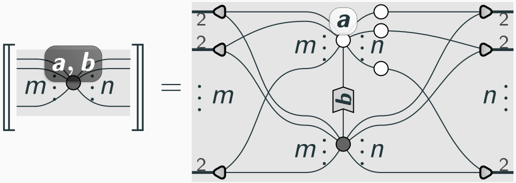
Lemma
There is an embedding $\mathsf{AffRel}_{\mathbb{K}} \to \mathsf{AffLagRel}_{\mathbb{K}}$ *given*

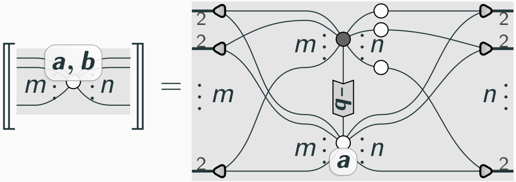
- **on objects by:** $n \mapsto 2n$;
- **on morphisms by:** $(S + \vec{a}) \mapsto S^{\perp} \oplus (S + \vec{a})$.

For the geometrically inclined, this is induced by the embedding of a vector space $\mathbb{R}^n \hookrightarrow T^*(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \oplus \mathbb{R}^n \cong \mathbb{R}^{2n}$ *into its cotangent bundle.*

Generators of affine Lagrangian relations (Comfort and Kissinger [CK22])

$\mathsf{AffLagRel}_{\mathbb{K}}$ is generated by two spiders decorated by $\mathbb{K}^2$; interpreted in $\mathsf{AffRel}_{\mathbb{K}}$ as:

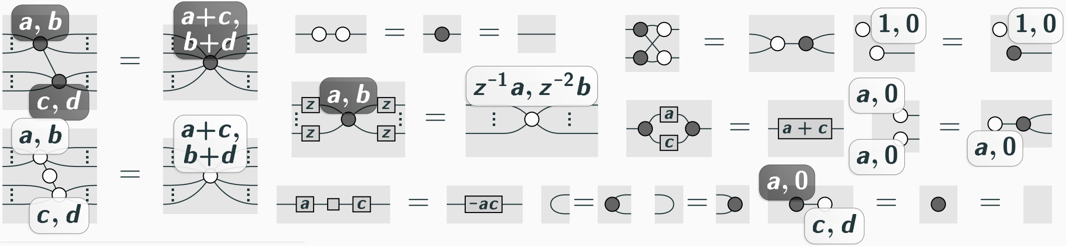




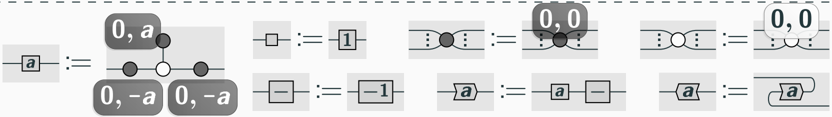
Equations of affine Lagrangian relations (Booth et al. [BCC24b])

Modulo both spiders, being commutative, undirected nodes,

as well as for all $a, b, c, d \in \mathbb{K}$ and $z \in \mathbb{K}^{\times}$:

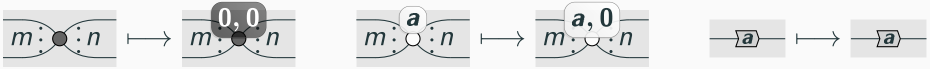


With derived generators:

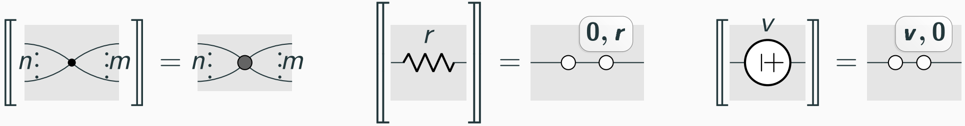


Interpreting electirical circuits

The embedding $\mathsf{AffRel}_{\mathbb{K}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{K}}$ takes:



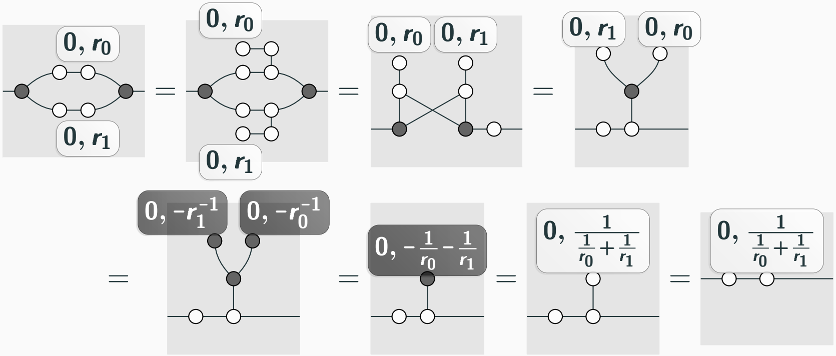
Now that the position/momentum wires are bundled together, we have a more concise description of electrical circuit components:



Example: composing resistors in parallel

$\mathrm{AffLagRel}_{\mathbb{R}}$ allows us to cleanly compose electrical circuits:

Example
Consider two resistors with resistances $r_0, r_1 \in \mathbb{R}^{>0}$ composed in parallel.



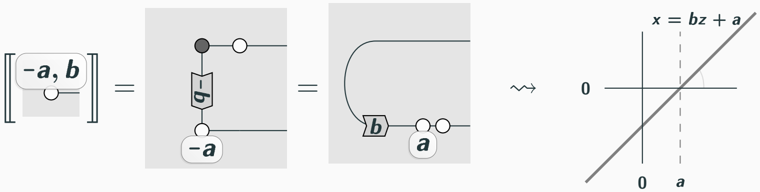
Electrons nondeterministically flow through both resistors, where they are impeded.

They extensionally behave like a resistor with resistance $1/(1/r_0 + 1/r_1)$.

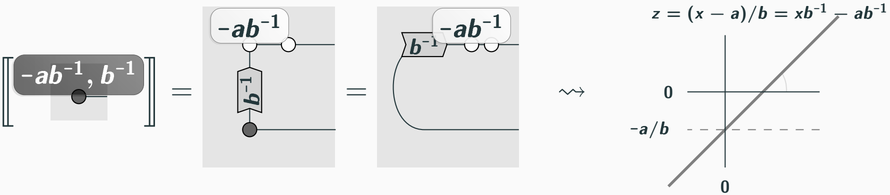
Interpreting the two spiders:

This colour-swap rule corresponds to a change of refrence frame.

Where configurations of phase space can be represented as functions of position:



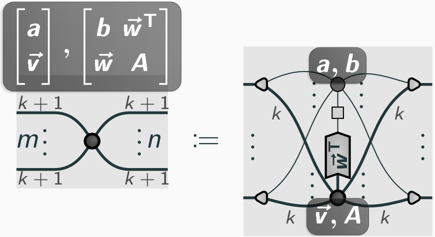
...or of momentum:

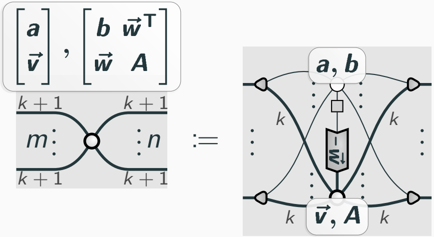


Scalable spiders

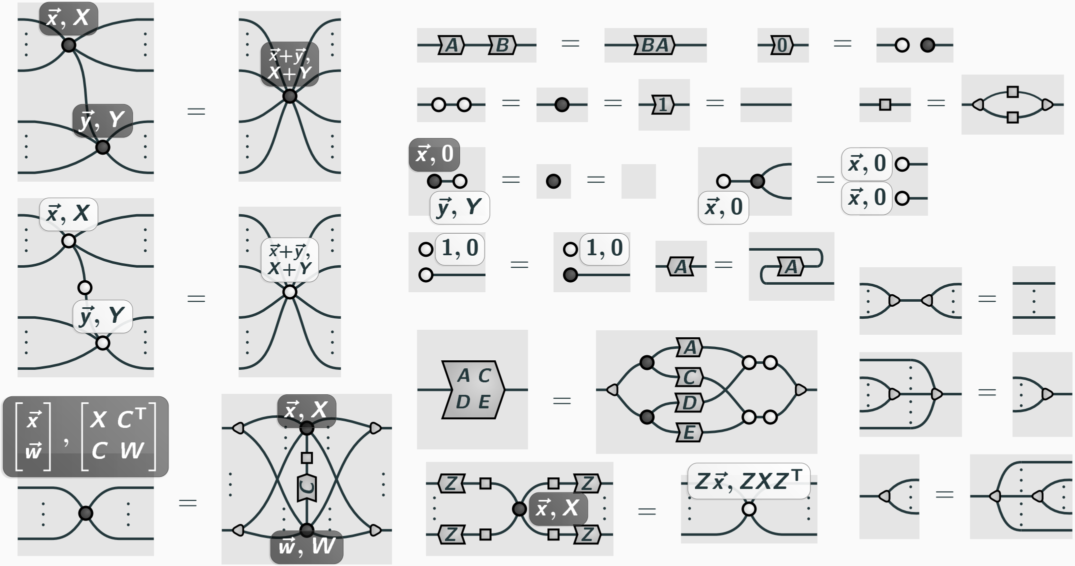
We can define higher-dimensional spiders by induction on the number of wires $k \in \mathbb{N}$.

Take $n, m \in \mathbb{N}$, $a, b \in \mathbb{K}$, $\vec{v}, \vec{w} \in \mathbb{K}^k$ and $A \in \mathrm{Sym}_k(\mathbb{K})$.





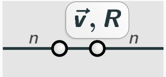
Scalable identities



Impedance matrix

Consider a network of resistors/voltage sources acting on n wires.

The extensional behaviour can be represented by a positive-definite $0 \prec R \in \mathrm{Sym}_n(\mathbb{R})$ called the **impedance matrix**, and a voltage $\vec{v} \in (\mathbb{R}^{>0})^n$



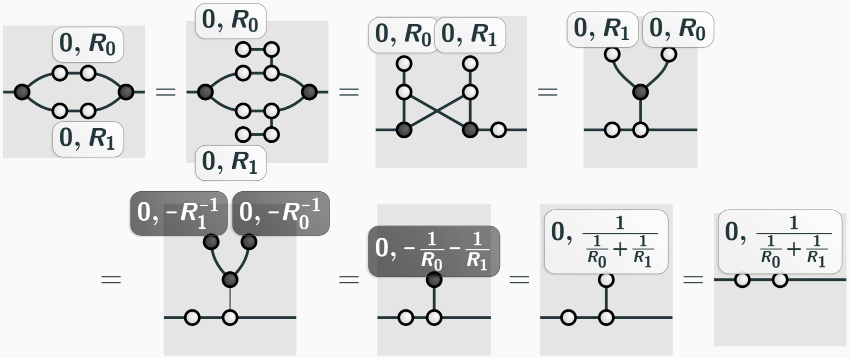
$$[\![\;\cdot\;]\!] = \left\{ \left(\begin{bmatrix} \vec{z} \\ \vec{x} \end{bmatrix}, \begin{bmatrix} \vec{z} \\ , \vec{x} + R\vec{z} + \vec{v} \end{bmatrix} \right) \mid \forall \vec{z}, \vec{x} \in \mathbb{R}^n \right\}$$

The resistance between the jth and kth wire is $r_{j,k} = r_{k,j} \in \mathbb{R}$.

The change in voltage on wire j is $v_j \in \mathbb{R}$.

Composing networks of resistors in parallel

Black-boxed networks of resistors compose in parallel in the same way as single resistors composed in parallel:



We don't know the internal structure of the two networks, but we still can compute their extensional behaviour in parallel.

Part II:
Complete equational theories for classical and quantum Gaussian relations

Cole Comfort

Joint work with: Robert I. Booth, Titouan Carette

based on: arXiv:2403.10479

June 20, 2024

Recap

Recall that the phase-space on n particles in Euclidean space is the symplectic vector spaces $(\mathbb{R}^{2n} \cong (\mathbb{R}^n)_Z \oplus (\mathbb{R}^n)_X, \omega_n)$:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure mom'um	pressure
Thermal	entropy	entropy flow	temperature mom'um	temperature

Where (maximally compatible, affinely constrained) mechanical circuits can be represented by string diagrams for $\mathsf{AffLagRel}_{\mathbb{R}}$.

Quantized fragments of quantum mechanics

"Quantized fragments" of quantum mechanics admit nondeterministic phase-space semantics:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure mom'um	pressure
Thermal	entropy	entropy flow	temperature mom'um	temperature

Quantized mechanics	Z	dZ/dt	X	dX/dt
Stabiliser QM *(finite dimensional)*	Pauli Z	Pauli Z flow	Pauli X	Pauli X flow
Gaussian QM *(infinite dimensional)*	$\hat{q}$	$\hat{q}$ flow	$\hat{p}$	$\hat{p}$ flow

Stabiliser quantum mechanics

Finite dimensional quantum mechanics “lives in” $(\mathrm{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$...

Definition
Fix some odd prime p. The state space of a **quopit** is the p-dimensional vector space:

$$\mathcal{H}_d := \ell^2(\mathbb{Z}/d\mathbb{Z}) = \mathrm{span}_{\mathbb{C}}\{|0\rangle, \cdots, |d-1\rangle\}$$

Definition
The n-quopit **Pauli group** $\mathcal{P}_p^{\otimes n} \subset \mathrm{U}(p^n)$ is generated under tensor product and composition by:

$$\mathcal{X}|k\rangle := |k+1\rangle \quad \text{and} \quad \mathcal{Z}|k\rangle := e^{i\frac{2\pi}{p}k}|k\rangle$$

Lemma
Because $\mathcal{X}\mathcal{Z} = e^{-i\frac{2\pi}{p}}\mathcal{Z}\mathcal{X}$ *every element of* $\mathcal{P}_p^{\otimes n}$ *has the following form,*

$$\chi(a)\mathcal{W}(\vec{z}, \vec{x}) := e^{i\frac{2\pi}{p}a}\bigotimes_{j=0}^{n-1}\mathcal{Z}^{z_j}\mathcal{X}^{x_j}$$

for some $a \in \mathbb{F}_p$, $\vec{z}, \vec{x} \in \mathbb{F}_p^n$.

Lemma
Up to scalars, a maximal Abelian subgroups $S \subseteq \mathcal{P}_p^{\otimes n}$ uniquely determines a normalised state $|S\rangle : \mathcal{H}_p^{\otimes n}$ such that for all $P \in S$, $P|S\rangle = |S\rangle$.

Such states are called **stabiliser states**.

Remark

Two n-quopit Pauli operators $\chi(a)\mathcal{W}(\vec{z},\vec{x})$ and $\chi(b)\mathcal{W}(\vec{q},\vec{p})$ commute if and only if $\omega_n((\vec{z},\vec{x}),\ (\vec{q},\vec{p})) = 0$.

Corollary (Gross [Gro06])
There is a bijection:

$$\begin{aligned}\{\textit{Maximal Abelian subgroups } S \subseteq \mathcal{P}_p^{\otimes n}\} &\cong \{\textit{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{F}_p^{2n}, \omega_n)\} \\ &\cong \{\textit{stabiliser states } |S\rangle : \mathcal{H}_p^{\otimes n}\}\end{aligned}$$

Given a Pauli $\chi(a)\mathcal{W}(\vec{z},\vec{x}) \in S$:

- *$\vec{z}$ are the positions;* • *$\vec{x}$ are the momenta;* • *a is determined by the affine shift.*

Phase-space representation of stabiliser states

Using the compact-closed structure of $(\mathsf{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$:

Definition
The compact prop of quopit **stabiliser circuits** is generated under tensor and composition of the linear operators:

- All quopit stabiliser states $0 \to n$;
- Caps $|j\rangle \otimes |k\rangle \mapsto \delta_{i,j}$ of type $2 \to 0$;
- The cup $\sum_{j=0}^{p-1} |j\rangle \otimes |j\rangle$ is already a stabiliser state of type $0 \to 2$.

The composition of $\mathsf{AffLagRel}_{\mathbb{F}_p}$ agrees with that of in $\mathsf{FVect}_{\mathbb{C}}$:

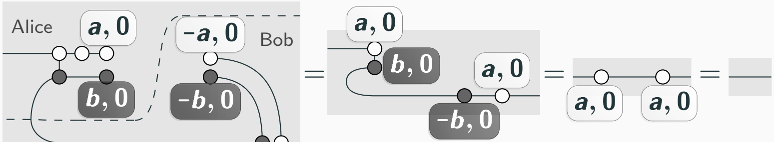
Theorem (Comfort and Kissinger [CK22])
$\mathsf{AffLagRel}_{\mathbb{R}_p}$ *isomorphic to quopit stabiliser circuits, modulo scalars.*

Remark

The presentation of $\mathsf{AffLagRel}_{\mathbb{F}_p}$ *is the stabiliser ZX-calculus of Poór et al. [Poó+23], modulo scalars.*

Picturing quantum teleportation

This is powerful enough to do quantum teleportation à la Abramsky and Coecke [AC04] and Coecke and Kissinger [CK18]:



Gaussian quantum mechanics

Continuous-variable quantum mechanics

Definition
The continuous-variable 1-D quantum state space is the Hilbert space:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x < \infty \right\}$$

The morphisms are bounded linear maps $(L^2(\mathbb{R}))^{\otimes n} \to (L^2(\mathbb{R}))^{\otimes m}$.

Definition
The **displacement** operators $\hat{Z}, \hat{X} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the CV-version of Paulis:

$$\hat{Z}(s) \circ \varphi(r) := e^{i2\pi rs}\varphi(r) \quad \text{and} \quad \hat{X}(s) \circ \varphi(r) := \varphi(r - s) \quad \text{for all} \quad r, s \in \mathbb{R},\ \varphi \in L^2(\mathbb{R})$$

The n-qumode **Heisenberg-Weyl group** $\mathcal{HW}^{\otimes n}$ is generated by displacement operators by tensor product and composition, where every Heisenberg-Weyl operator has the form:

$$\chi(a)\mathcal{W}(\vec{z}, \vec{x}) := e^{i2\pi a} \bigotimes_{j=0}^{n-1} \hat{Z}(z_j)\hat{X}(x_j)$$

The failure of CV stabilizer states

Lemma
Affine Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_n)$ *are in bijection with maximally Abelian subgroups of* $\mathcal{HW}^{\otimes n}$*, modulo scalars.*

Problem: *Given an affine Lagrangian subspace* $S \subseteq (\mathbb{R}, \omega_n)$*, there is no non-zero state* $|S\rangle : (L^2(\mathbb{R}))^{\otimes n}$ *such that* $\mathcal{W}(\vec{z}, \vec{x})\,|S\rangle$ *for all* $(\vec{z}, \vec{x}) \in \mathbb{R}^n$*!*

None of the states in $\mathsf{AffLagRel}_{\mathbb{R}}$ can be represented in Hilbert spaces!!!

$$\{\text{Maximal Abelian subgroups } S \subseteq \mathcal{HW}^{\otimes n}\} \cong \{\text{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{R}^{2n}, \omega_n)\}$$
$$\ncong \{\text{stabiliser states } |S\rangle : (L^2(\mathbb{R}))^{\otimes n}\}$$

Fixing the CV stabiliser formalism

Definition

An n-variate **Gaussian distribution** $\mathcal{N}(\Sigma, \vec{\mu})$ consists of a positive semidefinite covariance matrix $\Sigma \in \mathrm{Sym}_n(\mathbb{R})$ and a **mean** vector $\vec{\mu} \in \mathbb{R}^n$.

When Σ is positive-definite, $\mathcal{N}(\Sigma, \vec{\mu})$ admits a probability density function.

Proposition

A $2n$-variate Gaussian probability distribution $\mathcal{N}(\Sigma, \vec{\mu})$ on phase-space $(\mathbb{R}^{2n}, \omega_n)$ corresponds to a bounded state on $(L^2(\mathbb{R}))^{\otimes n}$ if and only if:

- Σ *is positive definite;*
- $\det(\Sigma) = 1$*;*
- $\Sigma + i \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ *is positive semidefinite.*

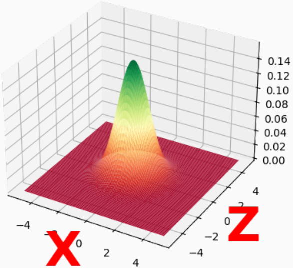
so that $\mathcal{N}(\Sigma, \vec{\mu})$ has a density function

respects Heisenberg's uncertainty principle for pure states

Call this a **quantum Gaussian distribution**.

Example

The **quantum vacuum state** $|0\rangle : L^2(\mathbb{R})$ is represented by the Gaussian distribution Φ_1 on $(\mathbb{R}^2, \omega_1)$:



$$\leftrightsquigarrow \mathcal{N}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0\\ 0\end{bmatrix}\right) =: \Phi_1$$

Φ_1 is the unique quantum Gaussian distribution on $(\mathbb{R}^2, \omega_1)$ invariant under rotation.

The Quantum Gaussian distribution Φ_n for $|0\rangle^{\otimes n}$ has the same universal property of being invariant under rotations (symplectic rotations $\mathrm{SO}(\mathbb{R}, 2n) \cap \mathrm{Sp}(\mathbb{R}, 2n)$).

Phase-space diagrams generated by Strawberry Fields/matplotlib

Explaining Heisenberg's uncertainty principle

The isomorphisms in $\mathsf{AffLagRel}_{\mathbb{K}}$ have the form:

Definition
An affine automorphism on $(\mathbb{K}^{2n}, \omega_n)$ is a **symplectomorphism** when it preserves the symplectic form.

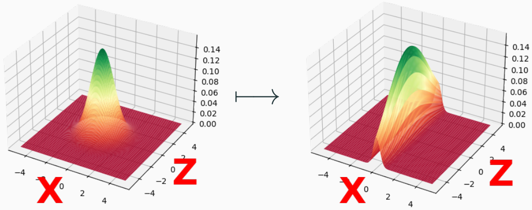
Lemma
Quantum Gaussian states are vacuum states acted on by affine symplectomorphisms.

Example
For $n = 1$, recall that $\omega_1 : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ measures area in $\mathbb{R}^2$.

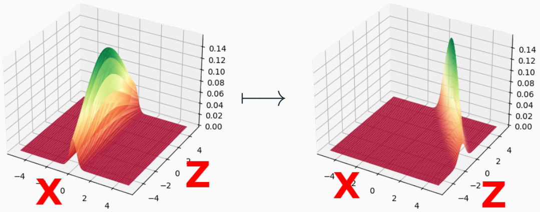
Therefore, quantum Gaussian states on $(\mathbb{R}^2, \omega_1)$ are generated by acting on the vacuum state with area-preserving affine isomorphisms.

Picturing area-preservation

For example, we can squeeze the Gaussian distribution for the vacuum state state:

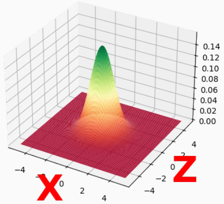


Changing the mean and rotating still is allowed.

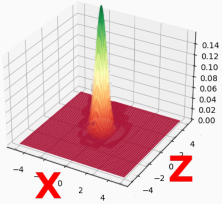


Picturing area-preservation

But we can not make Φ_1 more concentrated:



$\not\mapsto$



This violates Heisenberg's uncertainty principle.

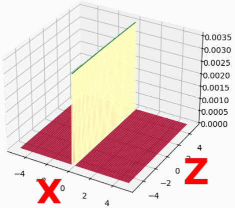
Approximating stabiliser states with Gaussian convolution

In phase-space CV stabiliser states do not have strictly positive definite covariance.

So they are not quantum Gaussian states.

However, they can be approximated with quantum Gaussian states:

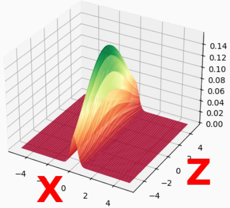
Dirac delta distribution



convolution by

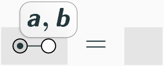
$$\xrightarrow{\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right)}$$

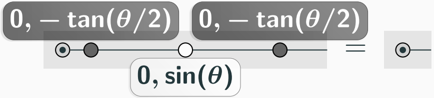
Gaussian density function



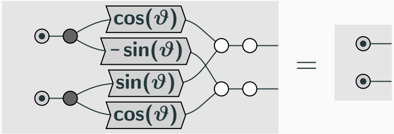
Because the vacuum state is the unique permissible Gaussian distribution in phase-space distribution invariant under rotation:

Theorem (Booth et al. [BCC24a])
The Gaussian state can be freely added to $\mathsf{AffLagRel}_{\mathbb{R}}$ *as a generator* ⊙–, *such that for all* $\vartheta \in [0, 2\pi)$ *and* $\theta \in (-\pi, \pi)$*:*





rotation of $\mathbb{R}^2$ *by* θ
decomposed into three shears



rotation of $\mathbb{R}^2$ *by* ϑ*, embedded along* $\mathsf{AffRel}_{\mathbb{R}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{R}}$

This contains both quantum Gaussian states and formal CV stabilisers.

Quantum Gaussian states and complexification

There is an equivalent formulation using the complex numbers

Proposition

Quantum Gaussian states/CV stabilisers can be represented by affine Lagrangian subspaces $S + \vec{a} \subseteq (\mathbb{C}^{2n}, \omega_n)$, *where:*

- $\vec{a}$ *is real;*
- *for all* $\vec{x} \in S$, $i\omega_n(\overline{\vec{x}}, \vec{x}) \geqslant 0$.

In other, words, we can represent the vacuum state as follows:

Theorem (Booth et al. [BCC24a])

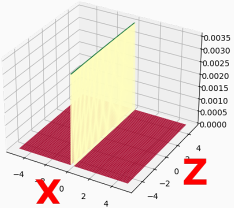
The Gaussian ZX-calculus is equivalent to adding the state 

to the image of the embedding $\mathrm{AffLagRel}_{\mathbb{R}} \hookrightarrow \mathrm{AffLagRel}_{\mathbb{C}}$.

Picturing Gaussian convolution

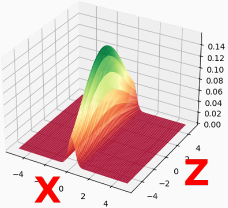
Dirac delta distribution

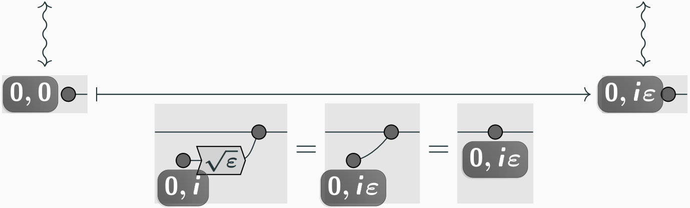
Gaussian density function



convolution by

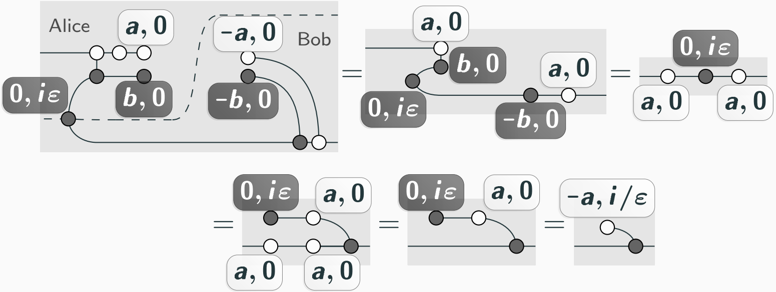
$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$$





Picturing continuous-variable quantum teleportation

We can interpret the continuous-variable quantum teleportation algorithm of Braunstein and Kimble [BK98]:



Fin

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