Graphical Calculi for Phase-Space Representations in Quantum Mechanics

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Outline

1. String diagrams
2. Generators and equations for affine relations
3. Quantum mechanics
4. Phase space and affine Lagrangian relations
5. Phase-space representation in finite-dimensional QM
6. Phase-space representation in *infinite*-dimensional QM
7. Research outlook: what remains to be done

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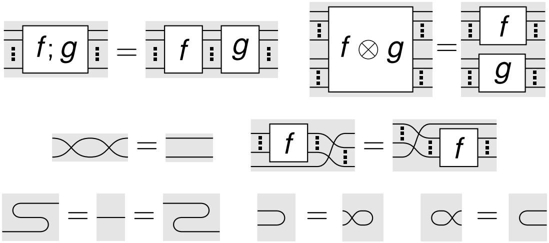
String diagrams

Compact closed categories have the syntax:



It is an ongoing area of research to find generators and equations for categories in terms of these string diagrams.

What parts of mathematics/physics/computer science can be reformulated in purely graphical terms?

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Affine relations

Given a field $\mathbb{K}$, *affine relations* over $\mathbb{K}$, $\mathsf{AffRel}_{\mathbb{K}}$ has:

Objects: natural numbers;

Maps: $n \to m$ are affine subspaces of $\mathbb{K}^n \oplus \mathbb{K}^m$;

Identity: $1_n := \{(\vec{v}, \vec{v}) \in \mathbb{K}^n \oplus \mathbb{K}^n\}$;

Composition: Given $S \subseteq \mathbb{K}^n \oplus \mathbb{K}^m$, $R \subseteq \mathbb{K}^m \oplus \mathbb{K}^k$,

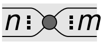
$$S; R := \{(\vec{v}, \vec{w}) \in \mathbb{K}^n \oplus \mathbb{K}^k \mid \exists \vec{u} \in \mathbb{K}^m, (\vec{v}, \vec{u}) \in S, (\vec{u}, \vec{w}) \in R\}$$

Compact closed structure:
Symmetric monoidal structure is pointwise direct sum
Caps and cups are given by the relations

$$\left\{\left(*, \begin{bmatrix}\vec{v}\\\vec{v}\end{bmatrix}\right) \in \mathbb{K}^0 \oplus \mathbb{K}^{n+n}\right\} \quad \text{and} \quad \left\{\left(\begin{bmatrix}\vec{v}\\\vec{v}\end{bmatrix}, *\right) \in \mathbb{K}^{n+n} \oplus \mathbb{K}^0\right\}$$

Generators for Affine Relations

$\mathrm{AffRel}_{\mathbb{K}}$ is generated by *spiders* and *scalar multiplication*,



$$\left[\!\left[\cdot \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \left\{ \left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}, \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \right) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; a \in \mathbb{K} \right\}$$

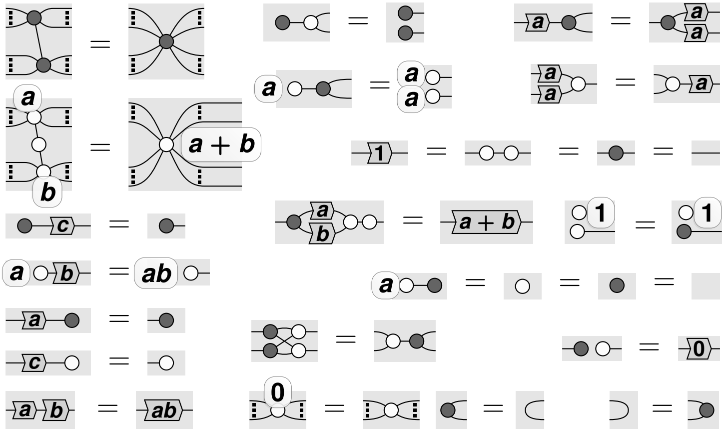
$$\left[\!\left[\cdot \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \left\{ (\vec{b}, \vec{c}) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; \sum_{j=0}^{n-1} b_j + \sum_{k=0}^{m-1} c_k = a \right\}$$

$$\left[\!\left[\cdot \right]\!\right]^{\mathrm{AR}}_{\mathrm{GAA}} := \{(b, ab) \mid b \in \mathbb{K}\}$$

for all $a \in \mathbb{K}$.

Equations for Affine Relations

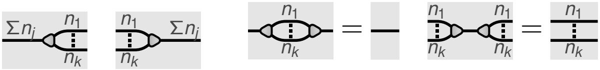
Mod the equations making spiders into undirected graphs and:



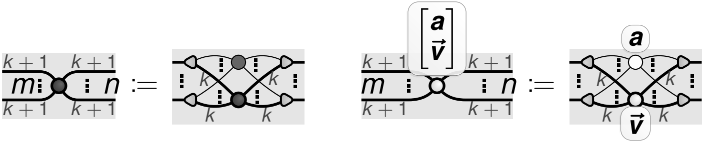
This was originally proved by [BPSZ19]. The original presentation is slightly different.

Strictification and block matrices

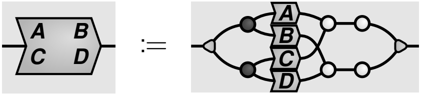
By working in the strictification of $\mathrm{AffRel}_{\mathbb{K}}$ we can bundle up multiple wires together (drawn thick):



Therefore we can define higher dimensional spiders:



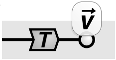
This gives us an inductive definition of block matrices



The Kernel and Image

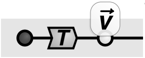
The strictification makes the normal forms easy to state.

Every affine subspace is the kernel of an affine matrix $(T, \vec{v}) : n \to m$:

⟦ 

⟧ $= \{(\vec{w}, *) \in \mathbb{K}^n \oplus \mathbb{K}^0 \mid T\vec{w} = \vec{v}\} \cong \ker(T, \vec{v})$

And similarly, for the image:

⟦ 

⟧ $= \{(*, T\vec{w} + \vec{v}) \mid \forall \vec{w} \in \mathbb{K}^n\} \cong \mathrm{im}(T, \vec{v})$

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Purely quantum mechanics in finite dimensions

Given some fixed $2 \leqslant d \in \mathbb{N}$, the state space for n-qudits is:

$$\mathcal{H}_d^{\otimes n} := (\mathrm{span}_{\mathbb{C}}\{|j\rangle\}_{j \in \mathbb{Z}/d\mathbb{Z}})^{\otimes n} \cong \mathbb{C}^{d^n}$$

$\mathcal{H}_d^{\otimes n}$ is interpreted as the state space for n-particles, each with d possible positions $|0\rangle, \cdots, |d-1\rangle$.

What position "means" here is a bit unintuitive...

An n-qudit quantum system is prepared and evolves as follows:

Pure states: normalized vectors in $\mathcal{H}_d^{\otimes n}$, ie:

$$\sum_{\vec{v} \in (\mathbb{Z}/d\mathbb{Z})^n} a_v |v\rangle \quad \text{st.} \quad \sum_{\vec{v} \in (\mathbb{Z}/d\mathbb{Z})^n} |a_v|^2 = 1$$

Quantum evolution: unitary operators $\mathcal{H}_d^{\otimes n} \to \mathcal{H}_d^{\otimes n}$

Unitaries preserve pure states.

Hilbert spaces

Pure qudit quantum mechanics lives in qudit Hilbert spaces:

Objects: generated by $\mathcal{H}_d^{\otimes n}$, for all $n \in \mathbb{N}$;

Maps: linear operators;

Monoidal product: *Bilinear tensor product*;

Compact structure: $1 \mapsto \sum_{j=0}^{d-1} |j\rangle\langle j| \quad \text{and} \quad |j\rangle\langle k| \mapsto \langle j|k\rangle$.

Dagger: Hermitian adjoint (complex conjugation)

Compact closure means that we can define linear maps

$$|k\rangle \mapsto \sum_j a_{j,k} |j\rangle$$

in terms of states:

$$\sum_{j,k} a_{j,k} |j\rangle\langle k|$$

Where $\{\langle j|\}$ is the dual basis of $\{|j\rangle\}$.

Quantum measurement: Born rule

A pure quantum state $|\varphi\rangle : \mathcal{H}$ can be represented by the rank-1 projector

$$|\varphi\rangle\langle\varphi| : \mathcal{H} \to \mathcal{H}$$

up to a global complex phase $e^{2\pi i a}$, for some $a \in [0.2\pi)$.

A **projection-valued measurement** on a state space $\mathcal{H}$ is defined with respect to an indexed orthonormal basis $B = \{\left|\psi_j\right\rangle\}_{j\in\mathcal{J}}$ for $\mathcal{H}$.

Measuring a state $|\varphi\rangle$ with respect to the basis B yields outcome j with probability $\left|\left\langle\psi_j\middle|\varphi\right\rangle\right|^2$.

Quantum measurement: mixed states

Measuring $|\varphi\rangle$ with respect to basis B can be reformulated in terms of applying the following operator to $|\varphi\rangle\langle\varphi|$:

$$\mathfrak{p}_B : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*;$$

$$|\varphi\rangle\langle\varphi| \mapsto \sum_{j \in \mathcal{J}} |\psi_j\rangle\langle\psi_j| \, |\varphi\rangle\langle\varphi| \, |\psi_j\rangle\langle\psi_j| = \sum_{j \in \mathcal{J}} \left|\langle\psi_j|\varphi\rangle\right|^2 |\psi_j\rangle\langle\psi_j|$$

These probabilistic mixtures of pure states are called **mixed quantum states.**

Formally adding probabilistic mixtures

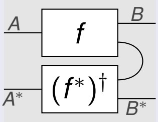
There is a category where measurement $\mathfrak{p}_B$ is a map:

Definition ([Sel07])

Given †-compact-closed C, CPM(C) has:

Objects are objects of C;

Maps $A \to B$ are given by maps $A \otimes A^* \to B \otimes B^*$ in C of the form:

$\mathrm{Tr}_X f :=$ 

for some object $X \in$ C and map $f : A \to B \otimes X$ in C.

†-compact closed structure pointwise in C.

Mixed quantum mechanics in finite dimensions

A map $f : A \to B$ in CPM(C) **trace-preserving** when $\mathrm{Tr}_B\, f = \mathrm{Tr}_A$.

Trace preserving states in CPM(FHilb) are mixed quantum states.

Trace-preserving maps preserve mixed quantum states.

In CPM(FHilb_d), we can prepare and evolve qudit systems using the two classes of operations:

Mixed quantum states: trace-preserving states $\mathcal{H}_d^{\otimes n}$;

Quantum evolution: trace-preserving maps $\mathcal{H}_d^{\otimes n} \to \mathcal{H}_d^{\otimes m}$.

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State space vs phase space

In quantum mechanics $\mathcal{H}_d^n$ is interpreted as a space of n particles with d possible positions.

Quantum states are described by probabilistic mixtures of normalized vectors in $\mathcal{H}_d^n$.

What if we regard states in terms of their **phase space**? Ie. the configurations of positions and *momenta*.

- How expressive is this?
- What are the categorical semantics?
- What is the unitary evolution?

Symplectic vector spaces i

A finite-dimensional vector space V is **symplectic** when it is equppied with an alternating, bilinear, non-degenerate bilinear form $\omega : V \oplus V \to \mathbb{K}$.
A **symplectomorphism** $T : (V, \omega) \to (V', \omega)$ is a linear isomorphism that preserves the symplectic form.

Theorem (Linear Darboux)

Every f.d symplectic vector spaces is symplectomorphic to $(\mathbb{K}^{2n}, \omega_n)$ *for some* $n \in \mathbb{N}$, *where*

$$\omega\left(\begin{bmatrix}\vec{z}\\ \vec{x}\end{bmatrix}, \begin{bmatrix}\vec{p}\\ \vec{q}\end{bmatrix}\right) := \vec{z}^{\mathsf{T}}\vec{q} - \vec{x}^{\mathsf{T}}\vec{p}$$

$(\mathbb{K}^{2n} \cong \mathbb{K}^n \oplus \mathbb{K}^n, \omega_n)$ is the phase-space of configurations of positions and momenta of n-particles.

Symplectomorphisms are unitary evolution.

Symplectic vector spaces ii

The **symplectic complement** of an affine subspace $S + \vec{a}$ of (V, ω) is:

$$S^\omega + \vec{a} := \left\{ \vec{v} \in V \,\middle|\, \forall \vec{s} \in S : \omega(\vec{v}, \vec{s}) = 0 \right\} + \vec{a} \subseteq V$$

An affine subspace $S + \vec{a}$ of a symplectic vector space (V, ω) is:

isotropic if $S \subseteq S^\omega$ (so that for all $\vec{s}, \vec{t} \in S$, $\omega(\vec{t}, \vec{s}) = 0$);

coisotropic if $S^\omega \subseteq S$;

Lagrangian if it is both isotropic and coisotropic ($S = S^\omega$).

The elements $(\vec{z}, \vec{x})$ of an affine Lagrangian subspace $S \subseteq (\mathbb{K}^{2n}, \omega_n)$ are interpreted as the *possible* configurations of abstract positions $\vec{z}$ and momenta $\vec{x}$ in a maximally constrained state S.

Affine Lagrangian relations

The compact prop of affine Lagrangian relations $\mathsf{AffLagRel}_{\mathbb{K}}$:

Objects: natural numbers;

Maps: $n \to m$ are affine Lagrangian subspaces of $(\mathbb{K}^{2n} \oplus \mathbb{K}^{2m}, \omega_{n,m})$, where:

$$\omega_{n,m}((\vec{v}_I, \vec{v}_O), (\vec{w}_I, \vec{w}_O)) := \omega_m(\vec{v}_O, \vec{w}_O) - \omega_n(\vec{v}_I, \vec{w}_I)$$

Composition/identities/monoidal structure: Same as in $\mathsf{AffRel}_{\mathbb{K}}$;

Compact structure:

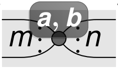
$$\left\{ \left(\begin{bmatrix} \vec{z} \\ \vec{z} \\ \vec{x} \\ -\vec{x} \end{bmatrix}, * \right) \in \mathbb{K}^{4n} \oplus \mathbb{K}^{0} \right\} \quad \text{and} \quad \left\{ \left(*, \begin{bmatrix} \vec{z} \\ \vec{z} \\ \vec{x} \\ -\vec{x} \end{bmatrix} \right) \in \mathbb{K}^{0} \oplus \mathbb{K}^{4n} \right\};$$

Dagger structure:

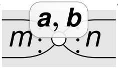
$$S^\dagger := \left\{ \left(\begin{bmatrix} \vec{z}_I \\ -\vec{x}_I \end{bmatrix}, \begin{bmatrix} \vec{z}_O \\ -\vec{x}_O \end{bmatrix} \right) \middle| \left(\begin{bmatrix} \vec{z}_I \\ \vec{x}_I \end{bmatrix}, \begin{bmatrix} \vec{z}_O \\ \vec{x}_O \end{bmatrix} \right) \in S \right\}.$$

Generators for affine Lagrangian relations

Given a field $\mathbb{K}$, $\mathsf{AffLagRel}_{\mathbb{K}}$ is generated by,



$$\left[\!\!\left[\; \cdot \; \right]\!\!\right]^{GSA}_{ALR} := \left\{ \left(\begin{bmatrix} \vec{z} \\ x \\ \vdots_m \\ x \end{bmatrix}, \begin{bmatrix} \vec{z'} \\ x \\ \vdots_n \\ x \end{bmatrix} \right) \middle| \; \vec{z} \in \mathbb{K}^m, \vec{z'}; \in \mathbb{K}^n, x \in \mathbb{K} : \sum_{j=0}^{m-1} z_j - \sum_{k=0}^{n-1} z'_k + bx = a \right\}$$

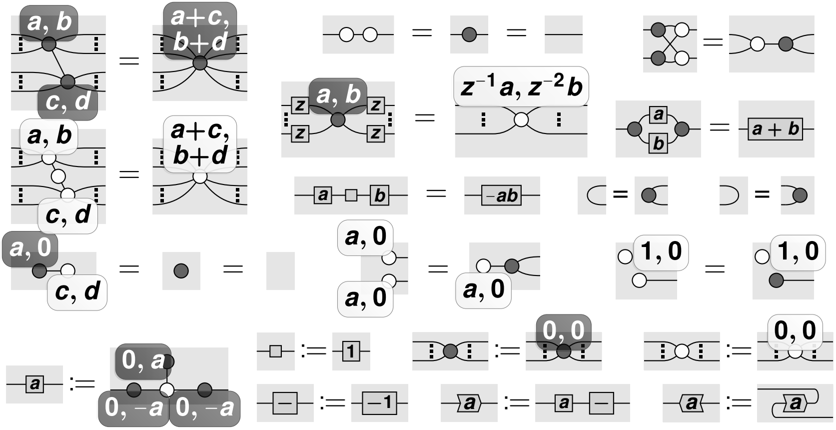


$$\left[\!\!\left[\; \cdot \; \right]\!\!\right]^{GSA}_{ALR} := \left\{ \left(\begin{bmatrix} z \\ \vdots_m \\ z \\ \vec{x} \end{bmatrix}, \begin{bmatrix} -z \\ \vdots_n \\ -z \\ \vec{x'} \end{bmatrix} \right) \middle| \; \vec{x} \in \mathbb{K}^m, \vec{x'} \in \mathbb{K}^n, z \in \mathbb{K} : \sum_{j=0}^{m-1} x_j + \sum_{k=0}^{n-1} x'_k - bz = a \right\}$$

for all $a, b \in \mathbb{K}$, $n, m \in \mathbb{N}$.

Equations for affine Lagrangian relations

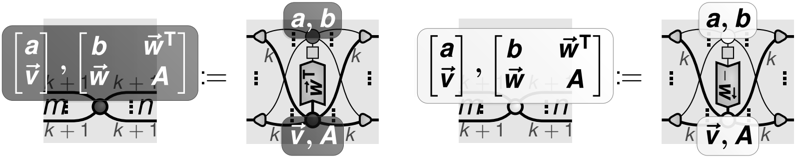
Modulo spiders being undirected, coloured graphs and,



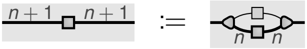
for all $a, b, c, d \in \mathbb{K}$, $z \in \mathbb{K}^{\times}$.

The strictification of affine Lagrangian relations

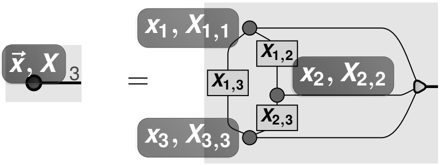
Define higher dimensional spiders, for $n, m \in \mathbb{N}$, $a, b \in \mathbb{K}$, $\vec{v}, \vec{w} \in \mathbb{K}^k$ and $A \in \mathrm{Sym}_k(\mathbb{K})$:



As well as the Fourier transform:



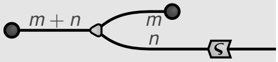
A k-coloured grey spider with n inputs and m outputs parametrizes an undirected coloured open graph.
For example, with $n = 0$, $m = 1$ and $k = 3$:



Normal form for affine Lagrangian relations

Every state in $\mathrm{AffLagRel}_{\mathbb{K}}$ is uniquely represented by a partially-open graph state:

$$\left[\begin{matrix}\vec{x} \\ 0_m \\ \vec{s}\end{matrix}\right], \left[\begin{matrix}0 & 1_m & F \\ 1_m & 0 & 0 \\ F^{\mathsf{T}} & 0 & S\end{matrix}\right]$$



for some $m \leqslant n \in \mathbb{N}$, $\vec{x} \in \mathbb{K}^m$, $\vec{s} \in \mathbb{K}^{n-m}$, $F \in \mathrm{M}_{m,n-m}(\mathbb{K})$ and $S \in \mathrm{Sym}_{n-m}(\mathbb{K})$, and permutation matrix $\varsigma \in \mathrm{M}_{n,n}(\mathbb{K})$.

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Phase-space representation in finite-dimensional quantum mechanics

For odd prime p, $\mathrm{AffLagRel}_{\mathbb{F}_p}$ is projective representation of pure **qupit stabilizer quantum circuits.**

Every affine Lagrangian relation $S + \vec{a} : 0 \to n$ is mapped to a **stabilizer state** $|S\rangle : \mathcal{H}_p^{\otimes n}$, up to global phase $\exp(2\pi i\alpha)$:

$$|S\rangle\langle S| := \frac{1}{p^n} \sum_{[\vec{z}^{\mathsf{T}}\ \vec{x}^{\mathsf{T}}]^{\mathsf{T}} \in S} \bigotimes_{j=0}^{n-1} \exp\big(2\pi i a_j/p\big) \exp\big(2\pi i z_j/p\big) \left|j + x_j\middle\rangle\middle\langle j\right|$$

In other words, up to scalars, $\mathrm{AffLagRel}_{\mathbb{F}_p}$ is a †-compact closed subcategory of FHilb.

Pure stabilizer quantum mechanics

Pure stabilizer evolution allows for two kinds of operations.

An *n*-qupit stabilizer quantum system has:

Pure states: stabilizer states on $\mathcal{H}_p^{\otimes n}$,
represented by affine Lagrangian subspaces of $({\mathbb{F}_p}^{2n}, \omega_n)$;

Quantum evolution: Clifford operators $\mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$,
represented by symplectomorphisms on $({\mathbb{F}_p}^{2n}, \omega_n)$.

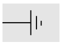
What about mixed states?

Phase-space representation of mixed states

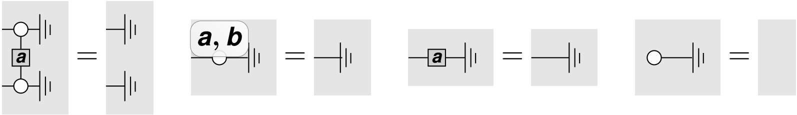
Theorem

$\mathrm{CPM}(\mathrm{AffLagRel}_{\mathbb{K}}) \cong \mathrm{AffCoIsotRel}_{\mathbb{K}}$

This is presented by adding a single generator interpreted as the discard relation:

$[\![$  $]\!] = \left\{ \left(\begin{bmatrix} z \\ x \end{bmatrix}, * \right) \in \mathbb{K}^2 \oplus \mathbb{K}^0 \right\}$

Modulo discarding of affine symplectomorphisms+states (isometries):



In $\mathrm{AffCoIsotRel}_{\mathbb{F}_p} \hookrightarrow \mathrm{CPM}(\mathrm{FHilb})$, this is interpreted as discarding a quantum state.

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Naïve phase space representation in infinite-dimensional quantum mechanics

The Hilbert space $L^2(\mathbb{R})$ is the state space of a **qumode** and $L^2(\mathbb{R}^n) \cong (L^2(\mathbb{R}))^{\otimes n}$ with the state space of *n*-qumodes:

$$L^2(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}^n} |f(\vec{v})|^2 \, d\vec{v} < \infty \right\}$$

The maps between Hilbert spaces are continous linear operators.

Affine symplectomorphisms on $\mathbb{R}^{2n}$ are a projective representation of **Gaussian unitaries** between *n*-qumodes.

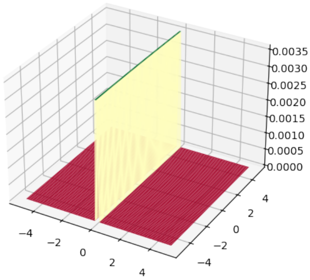
Projection onto real affine Lagrangian subspaces aren't continuous.
Eg, an affine Lagrangian subspace of $(\mathbb{R}^2, \omega_1)$ is a *line*!

Intuition: Gaussian convolution

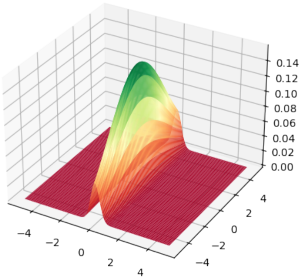
We want to add Gaussian noise to smooth things out:

Dirac delta "distribution" at $x = 0$

Gaussian density function



$\longmapsto$



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Gaussian probability theory

An n-variable **Gaussian distribution** $\mathcal{N}(\Sigma, \vec{\mu})$ is a probability distribution on $\mathbb{R}^n$ determined by some $0 \preceq \Sigma \in \mathsf{Sym}_n(n)$, called the **covariance matrix** and a vector $\vec{\mu} \in \mathbb{R}^n$, called the **mean.** The characteristic function of $\mathcal{N}(\Sigma, \vec{\mu})$ is given by

$$\vec{u} \mapsto \exp\left(i\vec{u}^{\mathsf{T}}\vec{\mu} - \frac{1}{2}\vec{u}^{\mathsf{T}}\Sigma\vec{u} \right)$$

Moreover, when $0 \prec \Sigma$, then $\mathcal{N}(\Sigma, \vec{\mu})$ has a density function given by

$$\vec{u} \mapsto \exp\left(\frac{-1}{2}(\vec{u} - \vec{\mu})^{\mathsf{T}}\Sigma^{-1}(\vec{u} - \vec{\mu}) \right) / \sqrt{(2\pi)^n \det(\Sigma)}$$

We perform Gaussian convolution on $\mathsf{AffLagRel}_{\mathbb{R}}$ to obtain a continuous variable phase-space semantics....

Gaussian quantum states

An n-qumode **Gaussian state** $\varphi \in L^2(\mathbb{R}^n)$ has the form:

$$\varphi(\vec{x}) = \exp(i\alpha)\exp\left(i\vec{s}^{\mathsf{T}}\vec{x}\right)\sqrt[4]{\det(\mathrm{Im}(\Phi))/\pi^n}\exp\left(i(\vec{x}-\vec{t})^{\mathsf{T}}\Phi(\vec{x}-\vec{t})/2\right)$$

where $\alpha \in [0, 2\pi)$, $\vec{s}, \vec{t} \in \mathbb{R}^n$, and $\Phi \in \mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Im}(\Phi) \succ 0$.
We call such a matrix Φ a **phase matrix**, and the vector $\begin{bmatrix} \vec{s}^{\,\mathsf{T}} & \vec{t}^{\,\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{2n}$ a **displacement**.

Together, they characterise the Gaussian state up to the "global phase" $\exp(i\alpha)$.

There is an important Gaussian state on $L^2(\mathbb{R})$ called the **vacuum** with trivial displacement and phase matrix i.

Wigner representation

The **Wigner transform** is a *real-valued* isomorphism $W_{(-)} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$:

$$W_\varphi \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} := \frac{1}{\pi^n} \int_{\mathbb{R}^n} \overline{\varphi}\left(\vec{q} + \vec{\xi}\right) \varphi\left(\vec{q} - \vec{\xi}\right) \exp\left(2i\vec{\xi}^{\mathsf{T}}\vec{p}\right)\, d\vec{\xi}$$

The Wigner transform of an *n*-qumode Gaussian state with phase matrix Φ and displacement $\vec{\mu}$ is the density function of the Gaussian distribution $\mathcal{N}(\Sigma, \vec{\mu})$ on $\mathbb{R}^{2n}$ with:

$$\Sigma := \begin{bmatrix} \operatorname{Im}(\Phi) + \operatorname{Re}(\Phi)\operatorname{Im}(\Phi)^{-1}\operatorname{Re}(\Phi) & -\operatorname{Re}(\Phi)\operatorname{Im}(\Phi)^{-1} \\ -\operatorname{Im}(\Phi)^{-1}\operatorname{Re}(\Phi) & \operatorname{Im}(\Phi)^{-1} \end{bmatrix}$$

Conversely, given a Gaussian distribution $\mathcal{N}(\Delta, \vec{\mu})$ on $\mathbb{R}^{2n}$ with,

$$\Delta := \begin{bmatrix} A & B \\ B^{\mathsf{T}} & C \end{bmatrix} \quad \text{with} \quad \det(\Delta) = 1 \quad \text{and} \quad \Delta + i\omega_n \succeq 0$$

there is a Gaussian state with $\Phi := -BC^{-1} + iC^{-1}$.

Phase-space representation of Gaussian QM

A complex affine Lagrangian relation $(S + \vec{a}) : n \to m$ is **positive** when for all $\vec{v} \in S$, $i\omega_{n,m}\left(\bar{\vec{v}}, \vec{v}\right) \geqslant 0$; and $\vec{a} \in \mathbb{R}^{2n}$.

Positive affine Lagrangian relations form a subcategory $\mathrm{AffLagRel}_{\mathbb{C}}^{+} \hookrightarrow \mathrm{AffLagRel}_{\mathbb{C}}$.

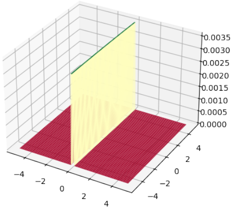
The Wigner representation of *n*-qumode Gaussian quantum states are positive affine Lagrangian relations $0 \to n$.

Positive affine Lagrangian relations are generated by adding shearing by i to $\mathrm{AffLagRel}_{\mathbb{R}}$, interpreted as the quantum vacuum state: 

Back to convolution

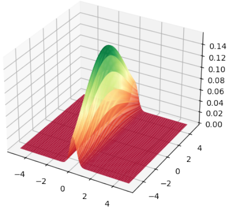
Dirac delta "distribution"

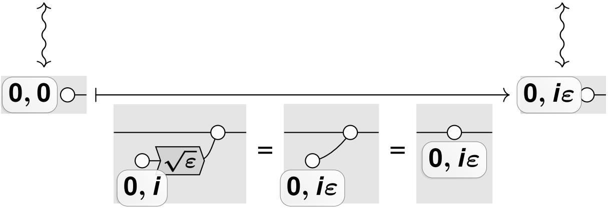
Gaussian density function



convolution by

$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) \longmapsto$$

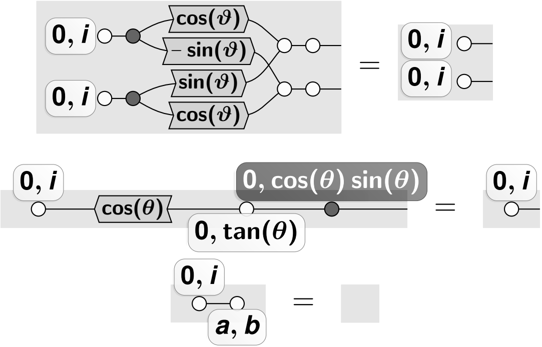




Generators and equations for Gaussian QM

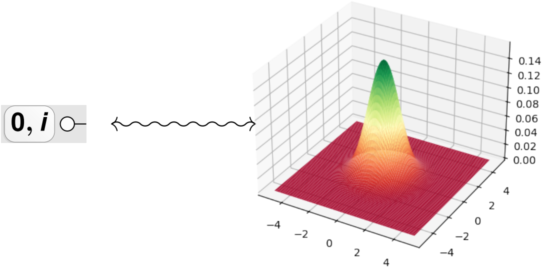
Syntactically, adding the vacuum is generated by freely codiscarding symplectic rotations $\mathrm{SO}(2n) \cap \mathrm{Sp}(2n)$ and effects in $\mathsf{AffLagRel}_{\mathbb{R}}$.

That is for all $a, b \in \mathbb{R}$, $\theta, \vartheta \in [0, 2\pi)$ with $\vartheta \notin \{\pi/2, 3\pi/2\}$:



Intuition for discarding

The vacuum Gaussian on 1-qumode is the standard bivariate normal distribution:

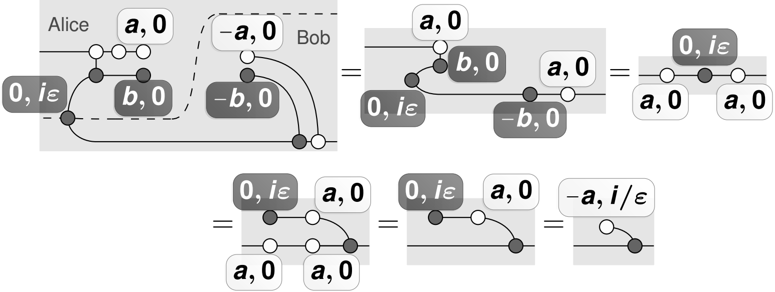


This is the only Wigner representation of a state which is invariant under rotation.

Higher dimensions harder to visualize.

Picturing quantum teleportation

Following [BK]:
Alice and Bob share a Gaussian Bell state with covariance of position $0 < \varepsilon \in \mathbb{R}$.
Alice records the homodyne measurement outcome $(a, b) \in \mathbb{R}^2$ in the Bell basis, and sends it to Bob,
who performs the phase correction $\hat{p}^{-b}\hat{q}^{-a}$:



The result is a quantum channel with an error; however, in the infinitely-squeezed limit of $\varepsilon = 0$ there is no error.

Outline

1. String diagrams
2. Generators and equations for affine relations
3. Quantum mechanics
4. Phase space and affine Lagrangian relations
5. Phase-space representation in finite-dimensional QM
6. Phase-space representation in *infinite*-dimensional QM
7. Research outlook: what remains to be done

Conclusion

We have turned the following Grassmanians into categories:
Affine Lagrangian/ coisotropic /positive Lagrangian

And given generators and relations+quantum interpretations.

What is next?

Orthogonal Grassmanian:
fermionic phase-space representation.
Lagrangian with respect to inner product.

Twisted affine coisotropic Grassmaninan:
Quantum dynamics $\mathbb{K}((s))$-affine subspaces.
Coisotropic with respect to Hermitian form
$\omega'_{n,m}(f(s), g(s)) := \omega_{n,m}(f(s), g(1/s))$

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