Notes on profunctors and compact multiplicative linear logic

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This document contains my ramblings about thinning links in MLL and the connection to the graphical calculus for pointed profunctors.

In [WGZ22] they reprove the coherence theorem for monoidal categories; constructing a strict monoidal category which is monoidally equivalent to a given monoidal category by inductively building up the strictification functor. Then they regard the coherence maps of the lax monoidal part of the functor as generators which we called tensor and unit introduction. The inverses to the coherence maps regarded as generators which we called cotensor and unit removal.

Very similar similar notation exists for proof nets for linearly distributive categories except the situation is more complicated because now there are two monoidal structures laxly "linearly" distributing over each other [BCST96]. Proof nets for linearly distributive categories are more nuanced than string diagrams for monoidal categories. One such nuance is the complication with units: extra virtual wires called "thinning links" are needed to unambiguously decide the connectivity of the units. Without thinning links then a proof net can be spuriously identified with distinct maps. Another problem is that one can draw diagrams that appear to be proof nets which are not valid in the sense that they do not correspond to a map in the linearly distributive category. For example one can form cycles using the graphical calculus which are not allowed. One way to tell if a proof net is valid is via the sequentialization procedure described in [BCST96, Section 2.4]. This is in some sense a linearly distributive analogue to the inductive strictification procedure for monoidal categories described in [WGZ22]; although it is not itself a strictification procedure. This involves inductively "boxing" a proof net, iterative adding the connectives and generators to the box. If the whole circuit is boxed when the procedure terminates, then one can conclude that the proof net is well-formed. This procedure was later reframed in terms of linearly distributive functor boxes in [CS99]. A linearly distributive functor (linear functor for short) consists of a lax monoidal functor and an oplax monoidal functor which distribute over each other appropriately. In their paper they establish a graphical calculus for lax and oplax monoidal functors to this end. In the graphical notation of [CS99], this is asymmetry is resolved with a "principle-port" on the side of the functor box where the wires are exiting from; so that the wires exiting from the principle port are bundled up and tensored together. Dually, an oplax monoidal functor is drawn with the principle port on the side of the box in which the wires are entering.

Note that monoidal categories are a particular case of linearly distributive categories where the linear distributor between both tensor products is the monoidal associator, so that both tensors are equal on the nose. There is still some subtlety here, linear functors betwen monoidal categories are not strong monoidal, they are instead Frobenius monoidal. In [CS18], they state without proof that thinning links are not needed for proof nets for monoidal categories, so that proof nets for monoidal categories are exactly the string diagrams in [WGZ22].

Functor boxes were independently rediscovered by [Mel06], where they were also applied to strong monoidal functors. The most aesthetically pleasing graphical calculus for lax/oplax/frobenius/strong monoidal functors in my opinion is contained in [McC12]. In this setting, they draw the "box" of a lax monoidal functor $F : \mathbb{X} \to \mathbb{Y}$ as string diagrams in $\mathbb{X}$ embedded withing the Poincaré dual of the shape generated by a monoid; this shape itself being embedded in the string diagrams for $\mathbb{Y}$. In this notation the principle port corresponds to where the wires exit the root of the Poincaré dual of a tree shape. For an oplax monoidal functor they do the same with a comonoid. For a strong monoidal functor they have shapes generated by a monoid and comonoid which are inverse to each other.

Therefore, take the any strictification of a monoidal category $S : \mathbb{X} \to \overline{\mathbb{X}}$. One would like to think that the strong monoidal functor boxes of [McC12] applied to S would reproduce exactly the result of [WGZ22]; however there is a snag. In [McC12], the monoid and its inverse which generate the shapes of the functor boxes are implicitly assumed to be strict. That is, they draw string diagrams for strict monoidal categories within the boxes, which breaks the whole premise.

There is a seemingly closely related graphical calculus called "internal string diagrams" [BDSPV15], although they explicitly state that it is a heuristic and is not formalized. In their setting they draw string diagrams which "look inside the points" of Vect-enriched profunctors. This looks very much like the functor box notation of [McC12]. Profunctors give an abstract notion of "shapes" where, for example, the the covariant and contravariant Yoneda embeddings of monoidal categories induce pseudomonoids and pseudocomonoids in profunctors. Looking inside the points of the shapes generated by these pseudomonoids and pseudocomonoids, one has has a map which has to conform to this shape. The internal string diagrams are thus drawn within the Poincaré dual of the string diagrams for the monoidal bicategory of profunctors. This was later rediscovered in the Set-enriched setting where they were called "open diagrams" [Rom20]. In [Rom20], they give an explicit description what is meant by "the points inside of profunctors" describing the category Prof_* of pointed profunctors. Although one must note that the this graphical calculus is still not fully worked out. Because we don't need Vect-enrichment here, will refer to this graphical calculus for Set-enriched profunctors unambiguously as the graphical calculus for pointed profunctors. Very recently, there has been more work towards formalizing these different notions of functor boxes [BR23].

The graphical calculus for pointed profunctors still doesn't make clear the connection to proof nets.

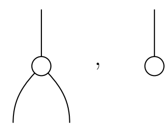
Inspired by the coherence via universality approach of Hermida [Her01, Her00], which I will touch on later, I wondered if there is a universal way to produce the strictification of a monoidal category in a way that simultaneously reproduces proof nets for monoidal categories. The appeal of a universal approach is that it would provide a general recipe to create string diagrams for other algebraic structures in monoidal categories. However, the universal approach of [Her00] is asymmetrical and needs to be adapted to reproduce proof nets for monoidal categories.

In the end, I didn't manage to succeed; however, I think that the route that I took in order to attempt this is insightful, and could potentially be useful to others.

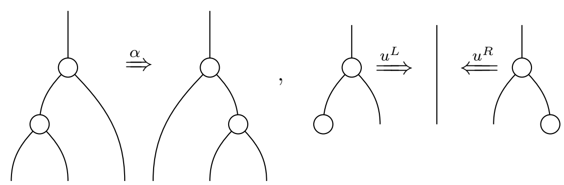
1 Pointed categories and multicategories

First, we categorify a monoid in a monoidal category:

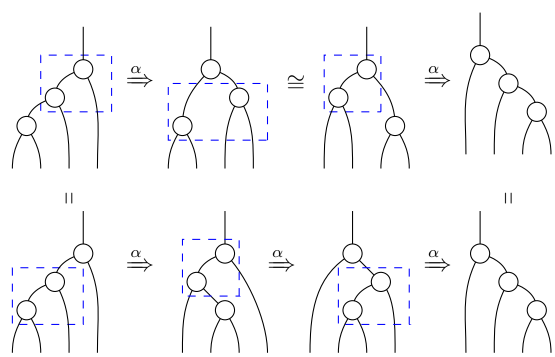
Definition .1. A **pseudomonoid** in a monoidal bicategory is an object $\mathcal{C}$ equipped with two 1-cells $C \otimes \mathcal{C} \to \mathcal{C}$ and $\mathbb{1} \to \mathcal{C}$ drawn as follows:



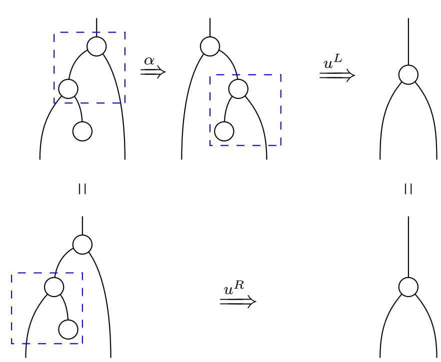
as well as three 2-cells, the associator, left and right unitors:



Satisfying the Mac Lane pentagon coherence equation (where the dashed blue box indicates where the nonidentity natural transformation is being applied):



As well as the unit coherences:



A pseudomonoid is **strict** when the associator and unitors are idenities.

If we take Cat to be a monoidal bicategory with respect to the Cartesian product, then monoidal categories have a slick definition:

Lemma .2. *Monoidal categories are pseudomonoids in* Cat *and strict monoidal categories are strict pseudomonoids in* Cat*.*

Contrast this with the view of small *strict* monoidal categories as categories internal to monoids between spans of sets; in the non-internal setting, the ability to express the pseudoness of the monoid allows the tensor product to not be strict.

By looking at the points inside of Cat, this way of viewing a monoidal category hints at some connection to proof nets:

Definition .3. The symmetric monoidal bicategory of **pointed categories** is the coslice bicategory $\mathsf{Cat}_* := \mathbb{1}/\mathsf{Cat}$. Explicitly, this has:

0-cells: Pointed categories, pairs consisting of a category along with a chosen object of that category:

$$(\mathbb{X}, X \in \mathbb{X}_0)$$

1-cells: Pointed functor between pointed categories, pairs consisting of a functor between the underlying categories and a morphism that preserves the point:

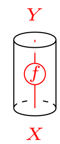
$$(F : \mathbb{X} \to \mathbb{Y}, f \in \mathbb{Y}(F(X), Y)) : (\mathbb{X}, X \in \mathbb{X}_0) \to (\mathbb{Y}, Y \in \mathbb{Y}_0)$$

2-cells: Given two parallel pointed functors $(F, f), (G, g) : (\mathbb{X}, X) \to (\mathbb{Y}, X)$, a pointed natural transformation is natural transformation $\varphi : F \to G$ that preserves the distinguished map, so that $\varphi_X : g = f$.

Composition of the 1-cells and 2-cells is given pointwise; and the monoidal structure is given by the Cartesian product.

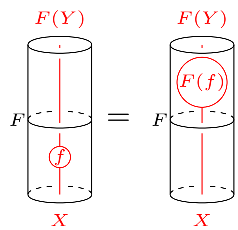
Recall the discussion about the graphical calculus for pointed profunctors. We will first restrict this graphical calculus to pointed categories and then return to pointed profunctors. We use the package in [BDSPV15] to typeset the diagrams.

Given a pointed functor, $(1_{\mathbb{X}}, f \in \mathbb{X}(1_{\mathbb{X}}(X), Y))$, draw the identity as a cylinder and the map inside the cylinder as follows:

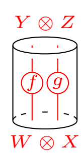


Think of the maps inside the category as "living" within the cylinder; wherein one can apply rewrite rules coming from the equational theory of the category.

Functors are drawn as membranes between separating the cylinder for the domain and codomain category; functoriality means that things can pass up through the membrane:



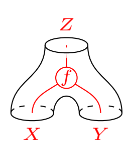
If $\mathbb{X}$ is a monoidal category, then one can tensor maps within the cobordism using the tensor product of $\mathbb{X}$, so that for $f : W \to X$ and $g : Y \to Z$, we have the the pointed functor $(1_{\mathbb{X}}, f \times \in \mathbb{X}(1_{\mathbb{X}}(W \otimes X), (Y \otimes Z)))$ with the following graphical representation:



Moreover, for every map with a binary tensor factorization of the domain $f : X \otimes Y \to Z$, we can use the external tensor product of Cat to obtain a pointed functor:

$$(\_ \otimes \_ : \mathbb{X}^2 \to \mathbb{X}, f \in \mathbb{Y}(X \otimes Y, Z)) : (\mathbb{X}^2, (X, Y) \in \mathbb{X}_0^2) \to (\mathbb{X}, Z \in \mathbb{X}_0)$$

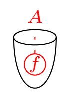
Drawn as follows:



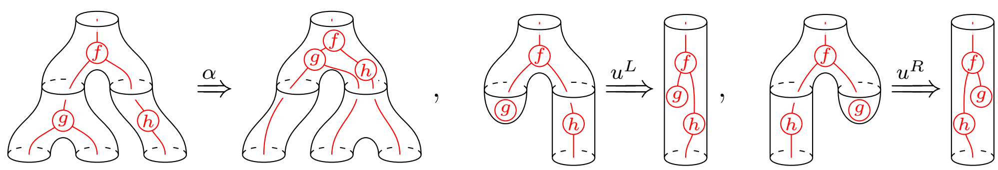
And for every state $f : I \to X$, there is a pointed functor

$$(I : \mathbb{1} \to \mathbb{X}, f \in \mathbb{Y}(I, X)) : (1, * \in \mathbb{1}) \to (\mathbb{X}, I \in \mathbb{X}_0)$$

Drawn as follows:



Consider the action of the associator and unitor on the points:



One way in which this diverges from proof nets is that these generators form a monoidal bicategory, not a monoidal category; forcing one to keep track of which 2-cells have been applied. A priori, in this setting, there is no guarantee that two different ways to get to the same diagram are equal.

This issue can be resolved via the multicategories. Informally, a multicategory is like a category, except for the multimaps now go from lists of objects in the domain to a single object in the codomain. Composition of multimaps corresponds to plugging a single output into an input, nesting trees. Strict monoidal categories correspond to the representable multicategories where every list of objects can be tensored together.

Internal multicategories can be constructed in a very similar way to internal categories (see [Lei04, Defininition 4.2.2] for a more general, thorough treatment):

Definition .4. $\mathsf{MultiSpan}$ is the bicategory with:

0-cells: Sets

1-cells:

$$\frac{[X] \xleftarrow{f} A \xrightarrow{g} Y \quad \text{in } \mathsf{Set}}{X \xrightarrow{(A,f)} Y \quad \text{in } \mathsf{MultiSpan}}$$

The identity on X is given by the span:

$$[X] \xleftarrow{\eta_X} X = X$$

The composition of multispans

$$X \xrightarrow{(f,A,g)} Y \xrightarrow{(h,B,k)} Z$$

is given by taking the following pullback

$$\begin{tikzcd}
 & & & {[A]_{[g]} \times_h B} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & & \\
 & & {[A]} \arrow[dl, "{[f]}"'] \arrow[dr, "{[g]}"] & & B \arrow[dl, "h"'] \arrow[ddrr, "k"] & & \\
 & {[[X]]} \arrow[dl, "\mu_X"'] & & {[Y]} & & & \\
{[X]} & & & & & & Z
\end{tikzcd}$$

Yielding a 1-cell:

$$X \xrightarrow{(\pi_0;[f];\mu_X,[A]_{[g]}\times_h B,\pi_1;k)} Z$$

Where, recall that $[-] : \mathsf{Set} \to \mathsf{Set}$ is the list monad where the unit $\eta_X(x) \mapsto [x]$ inserts into the singleton list and the multiplication μ flattens lists of lists.

2-cells: The 2-cell structure is the same as for spans and coherence 2-cells are essentially the same as for spans.

Definition .5. A (small) **multicategory** is a monad is $\mathsf{MultiSpan}$.

So we have a monad on the 1-cell $[\mathsf{Ob}] \xleftarrow{\mathsf{dom}} \mathsf{Ar} \xrightarrow{\mathsf{codom}} \mathsf{Ob}$. As opposed to the setting for internal categories, the domain is now a list of objects. The way that the composition and unit are defined plugs the single object in the codomain into an element in the list of objects in the domain.

An equivalent way to define a pseudomonoid in Cat would be to ask for a monoidal pseudofunctor $\mathbb{1} \to \mathsf{Cat}$ from the terminal monoidal category into Cat. Similarly a multifunctor from $\mathbb{1} \to \mathsf{Cat}$ yields what is sometimes called an unbiased monoidal category. This is very much like a monoidal category, except instead of there being a tensor product bifunctor, there is is a tensor product functor $\mathbb{X}^n \to \mathbb{X}$ for every arity $n \in \mathbb{N}$ which are compatible with each other.

Hermida [Her00], showed that when one regards an (unbiased) monoidal category as a multifunctor $F : \mathbb{1} \to \mathsf{Cat}$, by computing the following pullback in the bicategory of 2-multicategories, the Grothendieck category $\int F_{\mathbb{X}}$ is precisely a representable multicategory:

$$
\begin{tikzcd}
\int F \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathsf{Cat}_* \arrow[d] \\
\mathbb{1} \arrow[r, "F"'] & \mathsf{Cat}
\end{tikzcd}
$$

And moreover, the projection map $\pi_0 : F_{\mathbb{X}} \to \mathbb{1}$ is a fibration of multicategories. He shows that there is an equivalence of bicategories between the multifunctor category $[\mathbb{1}, \mathsf{Cat}]$ and the subcategory of the slice category of multifibrations over $\mathbb{1}$. In some sense, this equivalence can be regarded as an unbiased version of the coherence theorem for monoidal categories. This is quite elegant because strictification in this setting is a universal construction. However, from a string diagrammatic perspective, this is unsatisfying. Indeed, despite representable multicategories being in bijection with monoidal categories, the way that composition is defined in multicategories biases the inputs over the outputs; moreover, we are only allowed to compose along one object at a time.

2 Pointed profunctors and polycategories

To attempt to rectify this with the 2-sided nature of proof nets, we recall the category of internal profunctors described in Definition **??**, which is the 2-sided version of Cat. In this section it will be easier to work with profunctors *enriched* in Set, as opposed to profunctors internal to Set. This is more general as well, because it allows us to work with locally small categories, rather than merely small categories:

Definition .6. The category of Prof, of profunctors internal to Set has:

0-cells: Categories

1-cells: The morphisms are *profunctors* given by the following correspondence:

$$\frac{F : \mathbb{X}^{\mathsf{op}} \times \mathbb{Y} \to \mathsf{Set} \quad \in \ \mathsf{Cat}}{F : \mathbb{X} \nrightarrow \mathbb{Y} \quad \in \ \mathsf{Prof}}$$

The composition of profunctors $P : \mathbb{X} \nrightarrow \mathbb{Y}$ and $Q : \mathbb{Y} \nrightarrow \mathbb{Z}$ is given by the coend:

$$P;Q := \int^{Y\in\mathbb{Y}} P(-,Y)\times Q(Y,=) : \mathbb{X} \nrightarrow \mathbb{Z}$$

Where the coend of a functor $F : \mathbb{X}^{\mathsf{op}} \times \mathbb{X} \to \mathsf{Set}$ is given by the coequalizer diagram (in Cat):

$$\coprod_{X_1,X_2\in\mathbb{X}} \mathbb{X}(X_1,X_2)\times F(X_1,X_2) \rightrightarrows \prod_{X\in\mathbb{X}} F(X,X) \to \int^{X\in\mathbb{X}} F(X,X)$$

The identity profunctor on $\mathbb{X}$ is given by the hom functor

$$\mathbb{X}(-,=) : \mathbb{X}^{\mathsf{op}}\times\mathbb{X} \to \mathsf{Set}$$

Intutitively, this is a categorification of trace of a matrix where the natural numbers are replaced with categories, and the commutative ring is replaced with Set. Thus, the composition of profunctors categorifies matrix multiplication.

2-cells: 2-cells between parallel profunctors $P, Q : \mathbb{X} \nrightarrow \mathbb{Y}$ are natural transformations between the underlying profunctors $P, Q : \mathbb{X}^{\mathsf{op}} \times \mathbb{Y} \to \mathsf{Set}$.

Compact closed structure: The symmetric monoidal structure of Prof is given by extending the Cartesian structure in Cat. The units and counits of the compact closed structure are given by the hom functor.

A comprehensive review of the basic theory of profunctors and their coend calculus is contained in [Lor21]. There are two classes of profunctors which will be of interest to us:

Definition .7. A profunctor $\mathbb{X} \nrightarrow \mathbb{Y}$ is **representable** when it is naturally isomorphic to the profunctor $F_* := \mathbb{Y}(F-,=)$ for a functor $F : \mathbb{X} \to \mathbb{Y}$.

Dually, a profunctor $\mathbb{Y} \nrightarrow \mathbb{X}$ is **corepresentable** when it is naturally isomorphic to the profunctor $F^* := \mathbb{Y}(-,F=)$ for a functor $F : \mathbb{X} \to \mathbb{Y}$.

There are two embeddings of Cat into Prof that preserve the monoidal structure (formally, they are strong monoidal pseudofunctors):

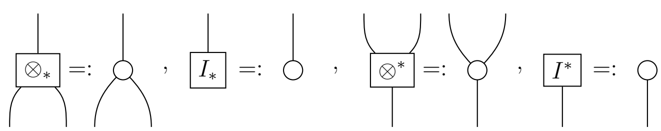
Definition .8. The representable embedding $(-)_* : \mathsf{Cat}^{\mathsf{co}} \to \mathsf{Prof}$ is the identity on objects, covariant on 1-cells and contravariant on 2-cells. It takes functors $F : \mathbb{X} \to \mathbb{Y}$ to profunctors $F_* : \mathbb{X} \nrightarrow \mathbb{Y}$.

Dually, the corepresentable embedding $(-)^* : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{Prof}$ is the identity on objects, contravariant on 1-cells and covariant on 2-cells. It takes functors $F : \mathbb{X} \to \mathbb{Y}$ to profunctors $F^* : \mathbb{Y} \nrightarrow \mathbb{X}$.

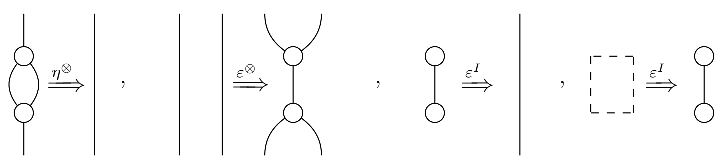
These two embeddings interact nicely:

Lemma .9. *Given any functor $F : \mathbb{X} \to \mathbb{Y}$, there is an adjunction $F_* \dashv F^*$ with unit η^F and counit ε^F.*

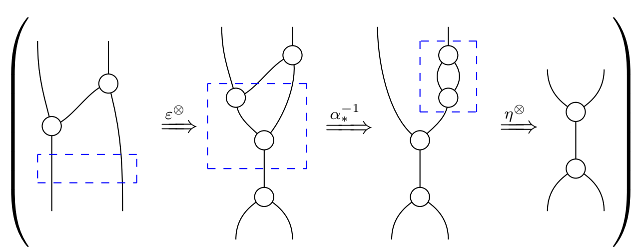
For example, if we draw the two embeddings of the pseudomonoid as follows:

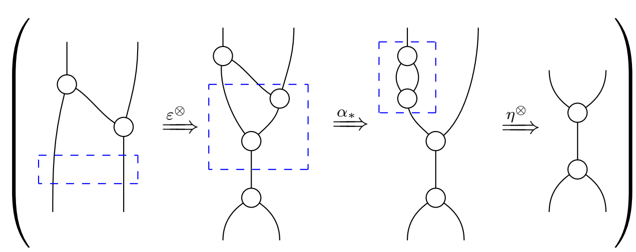


Then we have the following 2-cells:



Indeed for any monoidal category, the counit for the tensor product adjunction induces lax-Frobeniusators:

$\varphi^L :=$ 

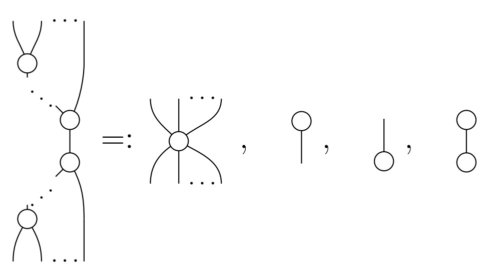
$\varphi^R :=$ 

These Frobeniusators interact with the (co)associators and (co)unitors of the (co)omonoid and adjoints to to satisfy several coherences forming what Franco et al call "map monoidal object" [FSW09, Remark 6.3]. They also remark that when the monoidal category is additionally autonomous, so that it has duals, the Frobeniusators are invertible. As a technical note, map monoidal objects in Prof are not precisely monoidal categories, as this biconditional only holds for Cauchy-complete categories.

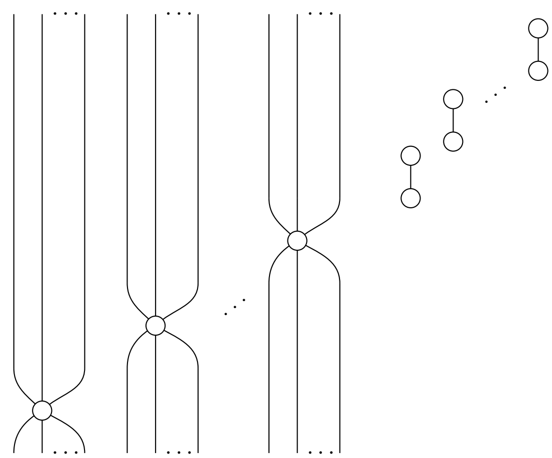
For context, a similar unpublished result Shulman discussed on the n-category cafe [Shu19] as well as in his paper [Shu20], characterizes biclosed linearly distributive categories and *-autonomous categories as, respectively, lax and pseudo-Frobenius algebras in the compact closed bicategory multivariable adjunctions. This refines the similar result of [Str04], where it is shown that Cauchy complete *-autonomous categories are in bijection with Frobenius pseudomonoids in Prof.

We can repeatedly apply natural transformations to reduce diagrams composed of the pseudo-Frobenius algbra structure coming from a monoidal category; however, to make things easier for us, from now on we will assume that the scalars are central, so that they commute with all maps in the monoidal category.

Definition .10. Fix a monoidal category $\mathbb{X}$ with central scalars. Say that a connected diagram in Prof composed of the generators of the corresponding pseudomonoid is in **spider normal form** when it is any of the four following types of diagrams (the last of which we will call a scalar spider):



Say that a not necessarily connected diagram is in **stratified spider normal form** when it can be composed into a strictly progressive sequence of nonscalar spiders followed by a strictly progressive sequence of scalar spiders:



Lemma .11. *Given a monoidal category $\mathbb{X}$, and a connected diagram in* Prof *generated by the* 1*-cells of the pseudo-Frobenius structure induced by $\mathbb{X}$ one can always reduce the diagram to spider normal form by repeated application of*

$$\varphi^L, \varphi^R, \eta^{\otimes}, \alpha_*, \alpha^*, (u^L)_*, (u^L)^*, (u^R)_*, (u^R)^*$$

As well as the symmetric monoidal structure of Prof*.*

Furthermore, given any not-necessarily connected *diagram can be reduced to stratified spider normal form using the same collection of* 2*-cells, where the order of the scalars spiders with respect to the tensor product is preserved by normalization.*

This lemma is an obvious corollary of spider theorem for special Frobenius algebras; however, the following is not so immediate:

Conjecture .12. The stratified spider normal form is strictly confluent so that any parallel 2-cells witnessing the reduction to the spider normal form are equal.

In some sense, it seems as if this should be seen to follow from Mac Lane's coherence theorem for monoidal categories [Lan78]; however, the author is unable to prove it. The rest of this chapter relies on this categorification of the spider theorem being true.

A similar result is proven for diagrams generated by the free pseudo-Frobenius algebra [DV19]. However, because the monoid and the comonoid are not required to be adjoint to each other, they require that the shapes be simply connected. They also require that the shape has a nontrivial boundary, so that scalars are not allowed.

Let us "look inside" Prof just as we did for Cat, to see how normalization acts on points.

Definition .13. The symmetric monoidal bicategory of **pointed profunctors**, Prof_* has:

0-cells: Pointed categories:

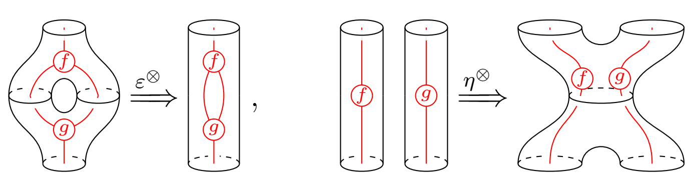
$$(\mathbb{X}, X \in \mathbb{X}_0)$$

1-cells: A pointed functor between pointed categories is a pair consisting of a profunctor between the underlying categories and a morphism that preserves the point:

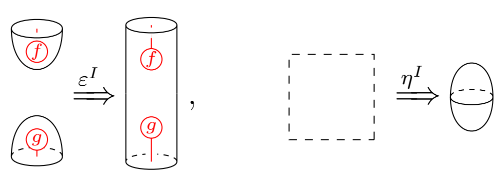
$$(F : \mathbb{X} \nrightarrow \mathbb{Y}, f \in F(X, Y)) : (\mathbb{X}, X \in \mathbb{X}_0) \to (\mathbb{Y}, Y \in \mathbb{Y}_0)$$

2-cells: Given two parallel pointed functors $(F, f), (G, g) : (\mathbb{X}, X) \to (\mathbb{Y}, Y)$, a pointed 2-cell is 2-cell $\varphi : F \Rightarrow G$ of profunctors that preserves the distinguished map, so that $\varphi_X : g = f$.

The graphical calculus for pointed profunctors is essentially the same as for pointed categories (recall that the graphical calculus for pointed categories which we used is already adapted from pointed profunctors). Except now, due to the two different Yoneda embeddings, we can factorize the domain and codomain of maps. For example, consider the action unit and counit for the tensor product:



And similarly for the tensor unit:

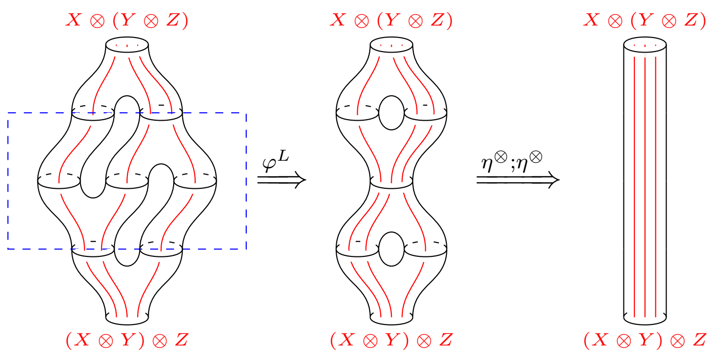


These diagrams have a striking resemblance to proof nets. One might naïvely try to obtain proof nets for monoidal categories in terms of quotienting the subcategory of Prof_* generated by the the monoidal structure by forcing the unit and counit of the adjunction to be equalities; however, the prospect of obtaining a monoidal category as a quotient of a monoidal bicategory is highly nontrivial.

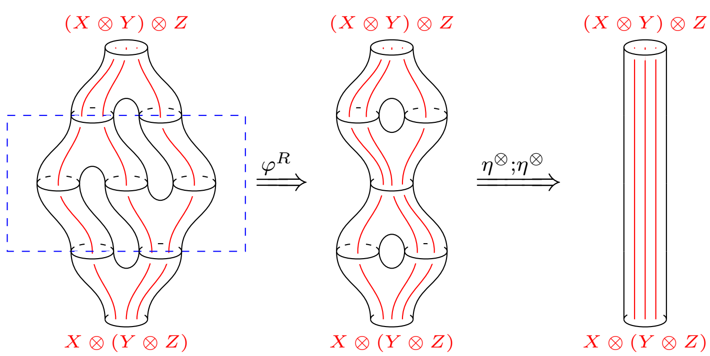
If we are more clever, we can obtain the associator regarded as an element of the hom-profunctor:

$$(\mathbb{X}(-,=), \alpha_{X,Y,Z}) : (\mathbb{X}(-,=), (X \otimes Y) \otimes Z) \nrightarrow (\mathbb{X}(-,=), X \otimes (Y \otimes Z))$$

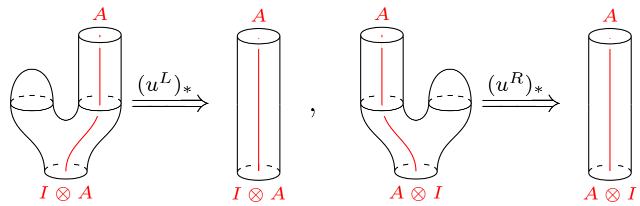
By normalizing this diagram:



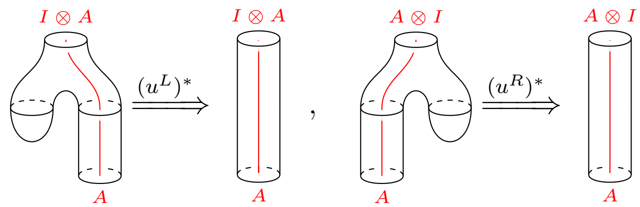
Similarly, we get the inverse associator by connecting the shapes in the other way:



We can do a similar thing for the left and right unitors:



As well as their inverses:



In the setting of proof nets, the tensor is inverse to the cotensor and the unit introduction is inverse to the unit removal.

3 Monoidal categories as displayed categories

The graphical calculus for Prof_* is slightly closer to proof nets than for Cat_* these 2-cells are at least adjoint. We want to turn the adjunctions into equalities somehow. We attempt this by regarding the normalization of the diagrams in Prof_* as the maps themselves. We need the following definition to this end:

Definition .14. A **displayed category** is an ordinary category $\mathcal{D}$ equipped with a lax normal functor $F : \mathcal{D} \to \mathsf{Prof}$. That is to say, F has the data of:

- A function $F : \mathcal{D}_0 \to \mathsf{Prof}_0$ taking objects of $\mathcal{D}$ to categories.
- For every pair of objects $X, Y \in \mathcal{D}_0$, a function $F_{X,Y} : \mathcal{D}(X,Y) \to \mathsf{Prof}(F(X), F(Y))$ such that $1_{F(X)} = F_{X,X}(1_X)$.
- For every triple of objects $X, Y, Z \in \mathcal{D}_0$, a 2-cell, with components at $(f : X \to Y, g : Y \to Z)$:

$$F_{X,Y,Z}(f,g) : F_{X,Y}(f); F_{Y,Z}(g) \Rightarrow F_{X,Z}(f;g)$$

Such that for any diagram $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{D}$ the following diagram commutes in Prof:

$$\begin{tikzcd}
(F_{W,X}(f); F_{X,Y}(g)); F_{Y,Z} \arrow[rr, "(\alpha_;)_{F_{W,X}(f), F_{X,Y}(g), F_{Y,Z}(h)}"] \arrow[d, "F_{W,X,Y}(f;g);1_{F_{Y,Z}(h)}"'] & & F_{W,X}(f); (F_{X,Y}(g); F_{Y,Z}(h)) \arrow[d, "1_{F_{W,X}(f)};F_{X,Y,Z}(g,h)"] \\
F_{W,Y}(f;g); F_{Y,Z}(h) \arrow[dr, "F_{W,Y,Z}(f;g,h)"'] & & F_{W,X}(f); F_{X,Z}(g;h) \arrow[dl, "F_{W,X,Z}(f,g;h)"] \\
& F(W,Z)(f;g;h) &
\end{tikzcd}$$

Where $\alpha_;$ is the associator for composition in Prof.

Let csfa denote the pro for special Frobenius algebras with central scalars. We can recast the conjectured spider theorem in this light:

Conjecture .15. Given any monoidal category $\mathbb{X}$ with central scalars, there is a displayed category $F_{\mathbb{X}} : \mathsf{csfa} \to \mathsf{Prof}$ such that:

- $(F_{\mathbb{X}})(n) = \mathbb{X}^n$
- $(F_{\mathbb{X}})n, m$ takes diagrams in csfa to (stratified) spiders in Prof.
- The natural transformation $(F_{\mathbb{X}})_{n,m,k}$ performs (stratified) spider fusion.

This is equivalent to the (conjectural) spider theorem. The confluence of the spider normalization corresponds to commutation of the pentagon in the definition of a displayed category.

This way of framing the categorified spider theorem lends itself to the following canonical construction, usually attributed to Bénabou [Bén72]. This is a variation of the eponymous construction of Grothendieck for pseudofunctors from ordinary categories into Cat, which we alluded to early in the multicategorical setting:

Theorem .16 (Bénabou-Grothendieck construction)**.** *Given a displayed category $F : \mathcal{D} \to \mathsf{Prof}$, the Bénabou-Grothendieck category, ΠF is given by the pullback:*

$$\begin{tikzcd}
\Pi F \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathsf{Prof}_* \arrow[d] \\
\mathcal{D} \arrow[r, "F"'] & \mathsf{Prof}
\end{tikzcd}$$

Where $\mathsf{Prof}_* \to \mathsf{Prof}$ is the canonical projection. Concretely ΠF has:

Objects: *Pairs* $(X \in \mathcal{D}_0, X^\sharp \in (F(X))_0)$

Maps: *The maps are pointed profunctors:*

$$(f, f^\sharp) : (X, X^\sharp) \to (Y, Y^\sharp)$$

where $f \in \mathcal{D}(X,Y)$ *and* $f^\sharp \in F_{X,Y}(f)(Y^\sharp, X^\sharp)$

Identity: $1_{(X,X^\sharp)} := (1_X, 1_{X^\sharp})$

Composition: *Given a composable pair:*

$$(X, X^\sharp) \xrightarrow{(f,f^\sharp)} (Y, Y^\sharp) \xrightarrow{(g,g^\sharp)} (Z, Z^\sharp)$$

The composite is defined as follows where $\widetilde{(f^\sharp, g^\sharp)}$ *is the canonical element induced by the coend on* $(f^\sharp, g^\sharp)$*:*

$$(f, f^\sharp); (g, g^\sharp) := (F_{X,Y,Z}(f,g))\widetilde{(f^\sharp, g^\sharp)}$$

Moreover, the first projection map $\pi_0 : \Pi F \to \mathcal{D}$ *is a (strict) functor.*

We can actually go the other direction. Given some fixed $\mathcal{D}$, this extends to an equivalence of categories between the slice category $\mathsf{Cat}/\mathcal{D}$ and the lax normal functor category $[\mathcal{D}, \mathsf{Prof}]$. Bénabou gives a detailed argument in proving this equivalence in his notes [Bén00]. We won't restate this equivalence of categories, as it takes considerable effort to expose.

Note that [BZ20] transported the work of Hermida on coherence via universality in the multicategorical setting to the *-polycategorical setting: where the inputs and outputs are now both lists of objects with duals. There is a similar construction of internal polycategories to internal categories and internal multicategories due to [Kos05]; however, it is quite a bit more complicated to describe formally. In [BZ20], they construct the Bénabou-Grothendieck **-polycatgory*, coming from a *displayed *-polycategory* $\mathbb{1} \to \mathsf{Prof}$. However, we can not directly appeal to this result if we want to recapture proof nets. This is because the composition in polycategories is defined only with respect to a single object at a time. So the counit for the tensor product doesn't even make sense because we can not form cycles.

The displayed *category* $F_{\mathbb{X}} : \mathsf{csfa} \to \mathsf{Prof}$ which we constructed is in some sense trying to recapture this way of looking at things. Let us see what happens when we compute the Bénabou-Grothendieck category of $F_{\mathbb{X}}$ (note that there is a monoidal version of the Cat-valued Grothendieck construction [MV20]; however we use the non-monoidal Prof-valued one because we need to two-sided nature of Prof):

Lemma .17. *The indexed category $\Pi F_{\mathbb{X}}$ has a concrete presentation:*

Objects: *Finite lists of objects in $\mathbb{X}$.*

Maps: *Given two finite lists $X = [X_0, \ldots, X_{n-1}]$ and $Y = [Y_0, \ldots, Y_{m-1}]$ of objects in $\mathbb{X}$, a map from $X \to Y$ is a pointed profunctor*

$$(P, f) : (\mathbb{X}^n, X) \nrightarrow (\mathbb{Y}^m, Y)$$

Generated by the (co)tensor and (co)unit of the monoidal structure of $\mathbb{X}$.

The equality of these maps is modulo the equivalence relation generated by the congruence $(P, f) \sim (Q, g)$ when Q and P are normalized to the same stratified spider $\nu_0 : P \Rightarrow S \Leftarrow Q : \nu_1$, where moreover, the normalization agrees on points so that $\nu_0(f) = \nu_1(g)$.

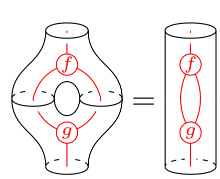
Identity: *The identity on $(\mathbb{X}^n, X)$ is the identity in pointed profunctors*

$$1_{(\mathbb{X}^n, X)} = (\mathbb{X}(-, =)^n, 1_X)$$

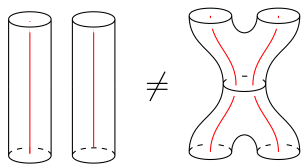
Composition: *The composition is the composition of pointed profunctors.*

This is a strict monoidal category. The tensor product is given by the tensor product in Prof_*, so that the projection functor $\pi_0 : \Pi F_{\mathbb{X}} \to \mathsf{csfa}$ is strict monoidal.

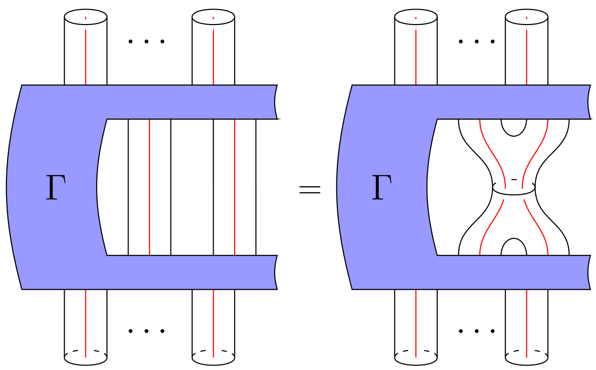
In $\Pi F_{\mathbb{X}}$, the components of the unitors and associators that we drew before as 2-cells are now merely maps which are inverse to each other. Moreover, $\eta^{\otimes}$ now induces the equality:



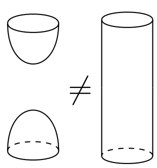
However, this is still not proof nets. We can only normalize connected components. For example, the following equation does not hold because the profunctors in which the string diagrams are drawn are not connected:



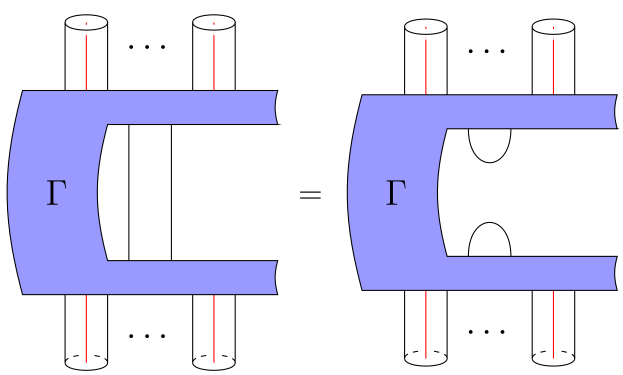
However, if in context Γ we knew that the profunctors were connected then we could apply this rewrite rule:



Similarly:



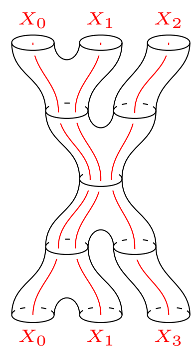
However, we know that the profunctors are connected in context Γ, we have:



To obtain proof nets, we would want all components to be connected.

In order to attempt this, remark that for any finite nonempty list of objects $X = [X_0, \ldots, X_{n-1}]$ in $\mathbb{X}$ there is an idempotent $(s_X, s_X^\sharp) = e_X : X \to X$ in $\Pi F_{\mathbb{X}}$ where s_X is the fully connected spider from $\mathbb{X}^n \to \mathbb{X}^n$ and $s_X^\sharp$ is the tensor factorized identity $s_X^\sharp = \otimes_{i=0}^{n-1} 1_{X_i}$ regarded as an element of s_X. Define $e_{[]} = (\mathbb{1}, *)$.

Consider the following example for the sake of illustration:

$e_{[X_0,X_1,X_2]} =$ 

We can regard these idempotents as the *property* that the boundary be connected, and obtain a new strict monoidal category by splitting them:

Definition .18. Take $N\mathbb{X} := \mathsf{Split}_{\{e_X \ | \ X \in [\mathbb{X}_0]\}}(\Pi F_{\mathbb{X}})$; that is, the full subcategory of the the Karoubi envelope of $\Pi F_{\mathbb{X}}$ with objects (X, e_X), for every finite list of objects in X. Concretely $N\mathbb{X}$ has:

Objects: Nonempty finite lists of objects in $\mathbb{X}$.

Maps: The maps $(P, f); X \to Y \in N\mathbb{X}$ are given by maps $(P, f) : X \to Y \in \Pi F_{\mathbb{X}}$ where the top boundaries of P are connected together, and the bottom boundaries are connected together.

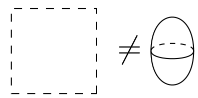
Composition: Same as in $\Pi F_{\mathbb{X}}$.

Identity: Same as in $\Pi F_{\mathbb{X}}$.

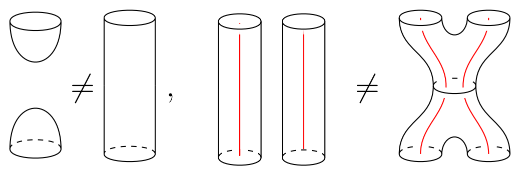
Monoidal structure: Given two $(P, f); W \to X$ and $(Q, g); Y \to Z$

$$(P, f) \otimes (Q, g) := e_{W,Y}; (P \times Q, (f, g)); e_{X,Z}$$

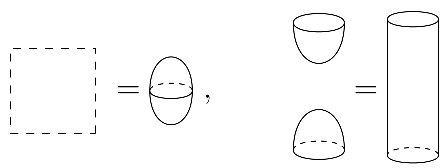
This monoidal category is very close to the strictification of a monoidal category; however, it is not quite there. There is no way to eliminate the scalars in this category. The following equation does not hold:



Moreover, the top and bottom boundaries need not be connected *to each other*, so we still don't recover the following desired equations:



If we additionally imposed the following equations as a quotient on $N\mathbb{X}$



this would yield a monoidal category in which all components are connected. This appears to recapture proof nets for monoidal categories in the setting where the scalars are central; however, the only reason this works is by direct appeal to the coherence result of [WGZ22]. This doesn't give any deep insight into why thinning links are not needed for monoidal categories, but are needed for linearly distributive categories.

The problems with units in *-autonomous categories and linearly distributive categories has long been known. For example, [Hou08] devotes his entire thesis to developing unitless multiplicative linear logic for this reason. However, it is disappointing that our naïve approach doesn't seem escape the quagmire of units in the degenerate case of monoidal categories where there is only one tensor product.

Nevertheless, I think there is a lot more to be done here; a proof of the uniqueness of the stratified spider normal form, or some variation of such, being the most important. It seems likely that that in order to precisely recapture proof nets for monoidal categories, we would have to change the setting in which we are computing the Bénabou-Grothendieck construction away from categories to some generalized version of polycategories where one can compose along multiple maps at the same time. In this setting, the category we are displaying over would no longer be scfa, everything would have to be connected. Then the analagous Bénabou-Grothendieck construction would connect all diagrams together.

The drawback of taking such approach, would be the difficulty of changing the displaying category. In the setting we have described, one could imagine that the displaying monoidal category is not only generated by a Frobenius algebra. For example, when one has two monoidal categories which are related by a strong monoidal functor, one would hope to adapt the indexing category to be generated by two Frobenius algebras on different objects along with a Frobenius algebra homorphism between them; then the total, monoidal category would have two different types of objects for each category glued together by the functor.

Perhaps the approach we have taken here, or some variation thereof is the correct level needed to glue proof nets of monoidal categories together. If we just forget about the units and change the indexing category to be generated by interacting non-unital, non-counital Frobenius algebras, the induced total monoidal category could potentially be useful for the concrete purpose of calculation: where one doesn't have to worry about keeping track of coherence information. This idea of glueing string diagrams for monoidal categories has been explored in [LZ23]; although they use the graphical calculus for pointed profunctors, so one has to keep track of coherence information.

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